Digitizing of special functions, differential equations, and solutions to Einstein's equations within a computer algebra system

Edgardo S. Cheb-Terrab

Physics, Differential Equations and Mathematical Functions, Maplesoft

Editor, Computer Physics Communications

Digitizing (old paradigm)

- ***Big amounts of knowledge available to everybody in local machines or through the internet***

- ***Take advantage of basic computer functionality, like searching and editing***

▼

Digitizing (new paradigm)

- ***By digitizing mathematical knowledge inside appropriate computational contexts that understand about the topics, one can***

  a) do an articulated, creative and useful use of this knowledge

  b) use the digitized knowledge to generate more and higher level knowledge

▼

Challenges

1) how to identify, test and organize the key blocks of information,

2) how to access it: the interface,

3) how to mathematically process it to obtain "derived information"

Three examples

Mathematical Functions

"Mathematical functions, are defined by algebraic expressions. So consider algebraic expressions in general ..."

The FunctionAdvisor (basic)

"Supporting information on definitions, identities, possible simplifications, integral forms, different types of series expansions, and mathematical properties in general"

Examples

```
> restart; FunctionAdvisor( )
```

```
The usage is as follows:
    > FunctionAdvisor( topic, function, ... );
where 'topic' indicates the subject on which advice is
required, 'function' is the name of a Maple function, and '.
..' represents possible additional input depending on the
'topic' chosen. To list the possible topics:
    > FunctionAdvisor( topics );
A short form usage,
    > FunctionAdvisor( function );
with just the name of the function is also available and
displays a summary of information about the function.
```

> *FunctionAdvisor*(*topics*)

```
The topics on which information is available are:
```

[*DE*, *analytic_extension*, *asymptotic_expansion*, *branch_cuts*, *branch_points*, *calling_sequence*, *class_members*, *classify_function*, *definition*, *describe*, *differentiation_rule*, *function_classes*, *identities*, *integral_form*, *known_functions*, *periodicity*, *plot*, *relate*, *required_assumptions*, *series*, *singularities*, *special_values*, *specialize*, *sum_form*, *symmetries*, *synonyms*, *table*] **(6.1.1.1)**

> *FunctionAdvisor*(*function_classes*, *quiet*)

[*trig*, *trigh*, *arctrig*, *arctrigh*, *elementary*, *GAMMA_related*, *Psi_related*, *Kelvin*, *Airy*, *Hankel*, *Bessel_related*, *0F1*, *orthogonal_polynomials*, *Ei_related*, *erf_related*, *Kummer*, *Whittaker*, *Cylinder*, *1F1*, *Elliptic_related*, *Legendre*, *Chebyshev*, *2F1*, *Lommel*, *Struve_related*, *hypergeometric*, *Jacobi_related*, *InverseJacobi_related*, *Elliptic_doubly_periodic*, *Weierstrass_related*, *Zeta_related*, *complex_components*, *piecewise_related*, *Other*, *Bell*, *Heun*, *trigall*, *arctrigall*, *integral_transforms*] **(6.1.1.2)**

> *FunctionAdvisor*(*bess*)

```
* Partial match of "bess" against topic "Bessel_related".
The 14 functions in the "Bessel_related" class are:
```

[AiryAi, AiryBi, BesselI, BesselJ, BesselK, BesselY, HankelH1, HankelH2, KelvinBei, KelvinBer, KelvinHei, KelvinHer, KelvinKei, KelvinKer] **(6.1.1.3)**

> *FunctionAdvisor*(*describe*, BesselK)

BesselK = *Modified Bessel function of the second kind* **(6.1.1.4)**

> *FunctionAdvisor*(*sum*, tan)

```
* Partial match of "sum" against topic "sum_form".
```

$$\left[\tan(z) = \sum_{\_kI=1}^{\infty} \frac{B_{2\_kI} (-1)^{\_kI} z^{-1+2\_kI} \left(4^{\_kI} - 16^{\_kI}\right)}{\Gamma(2\_kI+1)}, \textbf{ And } |z| < \frac{\pi}{2}\right] \quad \textbf{(6.1.1.5)}$$

> *FunctionAdvisor*(*integral*, Beta)

```
* Partial match of "integral" against topic
"integral_form".
```

$$\left[B(x,y) = \int_0^1 \_kI^{x-1} (1-\_kI)^{y-1} \, d\_kI, 0 < \Re(x) \textbf{ And } 0 < \Re(y)\right] \quad \textbf{(6.1.1.6)}$$

Summarize about a function

> *FunctionAdvisor*(GAMMA)

GAMMA

describe

Γ = *Gamma and incomplete Gamma functions*

definition

$\Gamma(z) = \int_0^{\infty} \frac{\_k1^{z-1}}{e^{\_k1}} \, d\_k1$	**And** $0 < \Re(z)$
$\Gamma(a, z) = \Gamma(a) - z^a \left(\int_0^1 \frac{\_t1^{a-1}}{e^{\_t1\, z}} \, d\_t1 \right)$	**And** $0 < \Re(a)$

analytic extension

$\Gamma(z) = \frac{\pi}{\sin(z\,\pi)\,\Gamma(1-z)}$	**And** $\Re(z) < 0$

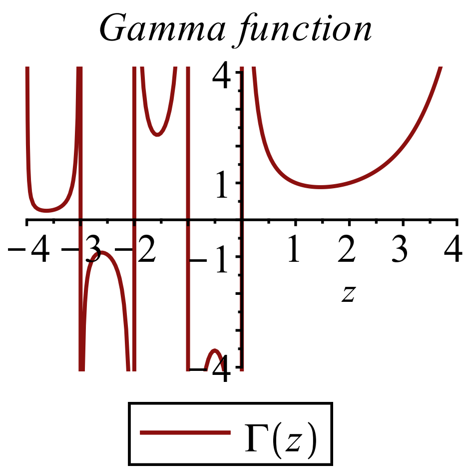
classify function

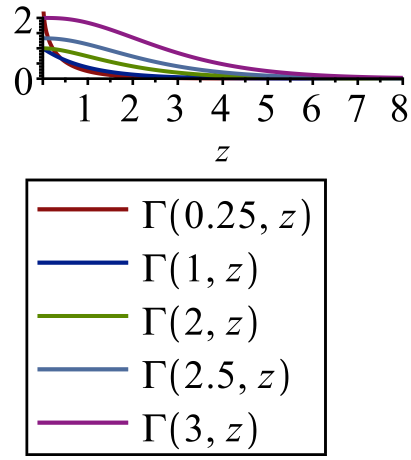
GAMMA_related	*1F1*

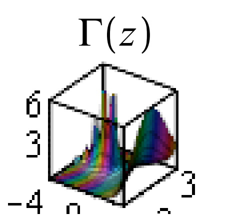
periodicity

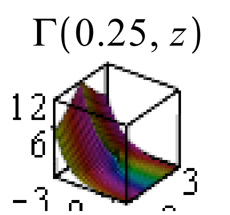
$\Gamma(z)$	No periodicity
$\Gamma(a, z)$	No periodicity

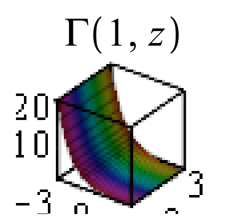
plot

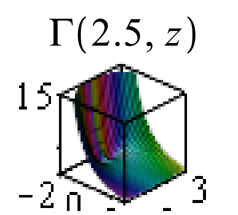












singularities

$\Gamma(z)$	z::*nonposint*	$z = \infty + \infty\,\mathrm{I}$
$\Gamma(a, z)$	$a = \infty + \infty\,\mathrm{I}$	$z = \infty + \infty\,\mathrm{I}$

branch points

$\Gamma(z)$	No branch points
$\Gamma(a, z)$	a::Not(*posint*) **And** $z \in [0, \infty + \infty\,\mathrm{I}]$

branch cuts

$\Gamma(z)$	No branch cuts
$\Gamma(a, z)$	a::Not($posint$) **And** $z < 0$

special values

$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(1) = 1$$

$$\Gamma(-1) = \infty + \infty\, \mathrm{I}$$

$$\Gamma(0) = \infty + \infty\, \mathrm{I}$$

$$\Gamma(\infty) = \infty$$

$$\Gamma(-\infty) = \mathit{undefined}$$

$$\Gamma(\infty\, \mathrm{I}) = 0$$

$$\Gamma(-\infty\, \mathrm{I}) = 0$$

$$\Gamma(-1, z) = \frac{\mathrm{Ei}_2(z)}{z}$$

$$\Gamma\left(-\frac{1}{2}, z\right) = -2\sqrt{\pi}\, \mathrm{erfc}\left(\sqrt{z}\right) + \frac{2\, \mathrm{e}^{-z}}{\sqrt{z}}$$

$$\Gamma(0, z) = \mathrm{Ei}_1(z)$$

$$\Gamma\left(\frac{1}{2}, z\right) = \sqrt{\pi}\, \mathrm{erfc}\left(\sqrt{z}\right)$$

$$\Gamma(1, z) = \mathrm{e}^{-z}$$

$$\Gamma(a, 0) = \Gamma(a)$$

$$\Gamma(a, \infty) = 0$$

$$\Gamma(a, -\infty) = \infty + \infty\, \mathrm{I}$$

identities

$$\Gamma(-z) = -\frac{\pi \csc(\pi z)}{\Gamma(z+1)}$$

$$\Gamma(z+y) = (z)_y \, \Gamma(z)$$

$$\Gamma(z) = \frac{2^{2z} \left(\prod_{\_k1=1}^{\infty} \frac{\Gamma\left(\frac{1}{2} + \frac{z}{2^{\_k1}}\right)}{\sqrt{\pi}} \right)}{z}$$

$$\Gamma(z) = 2^{\frac{3}{4} - z + \frac{\cos(2\pi z)}{4}} \pi^{\frac{\sin(\pi z)^2}{2}} (2z-2)!!$$

$\Gamma(a+n, z) = (a)_n \Gamma(a, z) + z^{a+n-1} e^{-z} \left(\sum_{\_k1=0}^{n-1} (1-a-n)_{\_k1} (-z)^{-\_k1} \right)$	**And** *n::nonnegint*
$\Gamma(a-n, z) = \frac{(-1)^n \Gamma(a, z)}{(1-a)_n} - e^{-z} z^{a-n} \left(\sum_{\_k1=0}^{n-1} 1 \Big/ (a-n)_{\_k1+1} z^{\_k1} \right)$	**And** *n::nonnegint*

sum form

$\Gamma(a,z) = \sum_{\_k1=0}^{\infty} \left(- \frac{z^{a} (-z)^{\_k1}}{\Gamma(\_k1+1)\,(\_k1+a)} \right) + \Gamma(a)$	**And** a::Not($nonposint$)
$\Gamma(a,z) = \left(\sum_{\_k1=0}^{-a-1} \frac{1}{\Gamma(1-a)\,e^{z}} \left(\Gamma(-a-\_k1)\, z^{\_k1+a}\, (-1)^{\_k1}\right) \right) + \left(\sum_{\_k1=0}^{\infty} \left(\left(\Psi(\_k1+1) - \ln(z)\right) z^{\_k1}\, (-1)^{-a+2\_k1}\right) \Big/ \left(e^{z}\,\Gamma(1-a)\,\Gamma(\_k1+1)\right) \right)$	a::$nonposint$ **And** $z \neq 0$
$\Gamma(a,z) = \sum_{\_k1=0}^{a-1} \frac{(a-1)!\, z^{\_k1}}{e^{z}\, \_k1!}$	a::$posint$ **And** $z \neq 0$

series

$$series(\Gamma(z), z, 4) = z^{-1} - \gamma + \left(\frac{\pi^2}{12} + \frac{\gamma^2}{2}\right) z + \left(-\frac{\zeta(3)}{3} - \frac{\pi^2 \gamma}{12} - \frac{\gamma^3}{6}\right) z^2 + \left(\frac{\pi^4}{160} + \frac{\zeta(3)\gamma}{3} + \frac{\pi^2 \gamma^2}{24} + \frac{\gamma^4}{24}\right) z^3 + O(z^4)$$

$$series(\Gamma(a, z), a, 4) = \mathrm{Ei}_1(z) + \left(\mathrm{Ei}_1(z)\ln(z) + G_{2,3}^{3,0}\left(z \middle| \begin{matrix}1, 1\\ 0, 0, 0\end{matrix}\right)\right) a + \left(\frac{\mathrm{Ei}_1(z)\ln(z)^2}{2} + \ln(z)\, G_{2,3}^{3,0}\left(z \middle| \begin{matrix}1, 1\\ 0, 0, 0\end{matrix}\right) + G_{3,4}^{4,0}\left(z \middle| \begin{matrix}1, 1, 1\\ 0, 0, 0, 0\end{matrix}\right)\right) a^2 + \left(\frac{\mathrm{Ei}_1(z)\ln(z)^3}{6} + \frac{\ln(z)^2\, G_{2,3}^{3,0}\left(z \middle| \begin{matrix}1, 1\\ 0, 0, 0\end{matrix}\right)}{2} + \ln(z)\, G_{3,4}^{4,0}\left(z \middle| \begin{matrix}1, 1, 1\\ 0, 0, 0, 0\end{matrix}\right) + G_{4,5}^{5,0}\left(z \middle| \begin{matrix}1, 1, 1, 1\\ 0, 0, 0, 0, 0\end{matrix}\right)\right) a^3 + O(a^4)$$

asymptotic expansion

$$asympt(\Gamma(z), z, 4) = \frac{1}{\left(\frac{1}{z}\right)^z e^z}\left(\sqrt{2}\sqrt{\pi}\sqrt{\frac{1}{z}} + \frac{\sqrt{2}\sqrt{\pi}\left(\frac{1}{z}\right)^{3/2}}{12} + \frac{\sqrt{2}\sqrt{\pi}\left(\frac{1}{z}\right)^{5/2}}{288} - \frac{139\sqrt{2}\sqrt{\pi}\left(\frac{1}{z}\right)^{7/2}}{51840} + O\left(\left(\frac{1}{z}\right)^{9/2}\right)\right)$$

$$asympt(\Gamma(a, z), a, 4) = \frac{1}{\left(\frac{1}{a}\right)^a e^a}\left(\sqrt{2}\sqrt{\pi}\sqrt{\frac{1}{a}} + \frac{\sqrt{2}\sqrt{\pi}\left(\frac{1}{a}\right)^{3/2}}{12} + \frac{\sqrt{2}\sqrt{\pi}\left(\frac{1}{a}\right)^{5/2}}{288} - \frac{139\sqrt{2}\sqrt{\pi}\left(\frac{1}{a}\right)^{7/2}}{51840} + O\left(\left(\frac{1}{a}\right)^{9/2}\right)\right)$$

integral form

$\Gamma(z) = \int_0^\infty \frac{\_k1^{z-1}}{e^{\_k1}} \, d\_k1$	**And** $0 < \Re(z)$
$\Gamma(a, z) = \Gamma(a) - z^a \left(\int_0^1 \frac{\_t1^{a-1}}{e^{\_t1 z}} \, d\_t1 \right)$	**And** $0 < \Re(a)$
$\Gamma(a, z) = \int_z^\infty \frac{\_k1^{a-1}}{e^{\_k1}} \, d\_k1$	**And** $0 < \Re(z)$

differentiation rule

$$\frac{\partial}{\partial a} \Gamma(a, z) = \Gamma(a, z) \ln(z) + G_{2,3}^{3,0}\left(z \,\middle|\, \begin{matrix} 1, 1 \\ 0, 0, a \end{matrix}\right)$$

$$\frac{\partial}{\partial z} \Gamma(a, z) = -z^{a-1} e^{-z}$$

$$\frac{\partial^n}{\partial z^n} \Gamma(a, z) = \frac{1}{a(-a+n-1)} \left(z^{-1-\frac{a}{2}+\frac{n}{2}} e^{-\frac{z}{2}} (a-n+1)_n \left((a+1) M_{\frac{a}{2}+\frac{n}{2}+1, \frac{a}{2}-\frac{n}{2}+\frac{1}{2}}(z) + (-n+z) M_{\frac{a}{2}+\frac{n}{2}, \frac{a}{2}-\frac{n}{2}+\frac{1}{2}}(z) \right) z^{a-n} \right) + \Gamma(a) (1-n)_n$$

DE

$f(z) = \Gamma(a, z)$	$\frac{d^2}{dz^2} f(z) = \frac{\left(\frac{d}{dz} f(z)\right)(a - z - 1)}{z}$

More complicated relationships between mathematical functions, computed using Maple internal database and algorithms.

> *FunctionAdvisor*(*specialize*, HermiteH, KummerU)

$$\left[H_a(z) = 2^a\, U\left(-\frac{a}{2}, \frac{1}{2}, z^2\right), \textbf{ And } 0 < \Re(z) \textbf{ or } \Re(z) = 0 \textbf{ And } 0 < \Im(z)\right] \quad \textbf{(6.1.1.7)}$$

> *FunctionAdvisor*(*DE*, EllipticF(*z*, *k*))

$$\left[f(z,k) = F(z,k), \left[\frac{\partial^2}{\partial k^2} f(z,k) = \frac{\left(-1 - 3k^4 z^2 + \left(z^2 + 3\right)k^2\right)\left(\frac{\partial}{\partial k} f(z,k)\right)}{k^5 z^2 + \left(-z^2 - 1\right)k^3 + k} + \frac{\left(z^3 - z\right)\left(\frac{\partial}{\partial z} f(z,k)\right)}{\left(k^4 - k^2\right)z^2 - k^2 + 1} + \frac{\left(-k^2 z^2 + 1\right) f(z,k)}{k^4 z^2 - k^2 z^2 - k^2 + 1}, \frac{\partial^2}{\partial k \partial z} f(z,k) = -\frac{\left(\frac{\partial}{\partial z} f(z,k)\right) k z^2}{k^2 z^2 - 1}, \frac{\partial^2}{\partial z^2} f(z,k) = \frac{\left(-2k^2 z^3 + \left(k^2 + 1\right) z\right)\left(\frac{\partial}{\partial z} f(z,k)\right)}{1 + k^2 z^4 + \left(-k^2 - 1\right) z^2}\right]\right] \quad \textbf{(6.1.1.8)}$$

> EllipticF(exp(sqrt(*z*)), *k*)

$$F\left(e^{\sqrt{z}}, k\right) \quad \textbf{(6.1.1.9)}$$

> *FunctionAdvisor*(*DE*, **(6.1.1.9)**, [*z*, *k*])

$$\left[f(z,k) = F\left(e^{\sqrt{z}}, k\right), \left[\frac{\partial^3}{\partial k^3} f(z,k) = \left(\frac{3\left(\frac{\partial^2}{\partial k \partial z} f(z,k)\right)}{\frac{\partial}{\partial z} f(z,k)}\right.\right.\right. \quad \textbf{(6.1.1.10)}$$

$$+ \frac{-5\,k^2+1}{k^3-k}\Bigg)\left(\frac{\partial^2}{\partial k^2}\,f(z,k)\right)+\left(\frac{(9\,k^2-3)\left(\frac{\partial}{\partial k}\,f(z,k)\right)}{(k^3-k)\left(\frac{\partial}{\partial z}\,f(z,k)\right)}\right.$$

$$+ \frac{3\,f(z,k)}{(k^2-1)\left(\frac{\partial}{\partial z}\,f(z,k)\right)}\Bigg)\left(\frac{\partial^2}{\partial k\partial z}\,f(z,k)\right)$$

$$+ \frac{(-4\,k^2-1)\left(\frac{\partial}{\partial k}\,f(z,k)\right)}{(k^2-1)\,k^2},\ \frac{\partial^2}{\partial z^2}\,f(z,k) = \Bigg(-k^3\,(k-1)^2\,(k+1)^2\left(\frac{\partial^2}{\partial k^2}\right.$$

$$\left.f(z,k)\right)^2 + \Bigg(\left(-6\,k^6+8\,k^4-2\,k^2\right)\left(\frac{\partial}{\partial k}\,f(z,k)\right)+2\,k\,(k-1)^2\,(k$$

$$+1)^2\left(\frac{\partial}{\partial z}\,f(z,k)\right)+\left(-2\,k^5+2\,k^3\right)f(z,k)\Bigg)\left(\frac{\partial^2}{\partial k^2}\,f(z,k)\right)$$

$$-9\left(k^2-\frac{1}{3}\right)^2 k\left(\frac{\partial}{\partial k}\,f(z,k)\right)^2+\Bigg(\left(6\,k^4-8\,k^2+2\right)\left(\frac{\partial}{\partial z}\,f(z,k)\right)$$

$$+\left(-6\,k^4+2\,k^2\right)f(z,k)\Bigg)\left(\frac{\partial}{\partial k}\,f(z,k)\right)+4\left(\frac{\partial}{\partial z}\,f(z,k)\right)^2 k\,z+\Big(2\,k^3$$

$$-2\,k\Big)\left(\frac{\partial}{\partial z}\,f(z,k)\right)f(z,k)-f(z,k)^2\,k^3\Bigg)\Bigg/\Bigg(-4\,k\,z\,(k-1)^2\,(k$$

$$+1)^2\left(\frac{\partial^2}{\partial k^2}\,f(z,k)\right)+\left(-12\,k^4\,z+16\,k^2\,z-4\,z\right)\left(\frac{\partial}{\partial k}\,f(z,k)\right)+\Big($$

$$-4\,k^3\,z+4\,k\,z\Big)f(z,k)\Bigg),\ \left(\frac{\partial^2}{\partial k\partial z}\,f(z,k)\right)^2 = \frac{k^2\left(\frac{\partial^2}{\partial k^2}\,f(z,k)\right)^2}{4\,z}$$

$$+\left(\frac{(3k^5-4k^3+k)\left(\frac{\partial}{\partial k}f(z,k)\right)}{2z(k-1)^2(k+1)^2}+\frac{(k^2-1)k^2 f(z,k)}{2z(k^4-2k^2+1)}\right)\left(\frac{\partial^2}{\partial k^2}f(z,k)\right)+\frac{(-2k^3+2k)\left(\frac{\partial}{\partial z}f(z,k)\right)\left(\frac{\partial^2}{\partial k\partial z}f(z,k)\right)}{k^4-2k^2+1}$$

$$+\frac{(3k^2-1)^2\left(\frac{\partial}{\partial k}f(z,k)\right)^2}{4z(k-1)^2(k+1)^2}+\frac{(3k^3-k)f(z,k)\left(\frac{\partial}{\partial k}f(z,k)\right)}{2z(k-1)^2(k+1)^2}$$

$$-\frac{\left(\frac{\partial}{\partial z}f(z,k)\right)^2k^2}{k^4-2k^2+1}+\frac{k^2f(z,k)^2}{4z(k-1)^2(k+1)^2}\Bigg]\Bigg]$$

The information returned by the FunctionAdvisor command can be used for further computations: verify the above

> *pdetest*(*op*(**(6.1.1.10)**))

$$[0, 0, 0] \tag{6.1.1.11}$$

The relation between all elementary functions and the pFq hypergeometric function:

> *FunctionAdvisor*(*elementary*, *quiet*)

[arccos, arccosh, arccot, arccoth, arccsc, arccsch, arcsec, arcsech, arcsin, arcsinh, arctan, arctanh, cos, cosh, cot, coth, csc, csch, exp, ln, sec, sech, sin, sinh, tan, tanh] **(6.1.1.12)**

> *map2*(*FunctionAdvisor*, *relate*, **(6.1.1.12)**, hypergeom)

$$\Bigg[\arccos(z)=\frac{\pi}{2}-z\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2\right),\ \text{arccosh}(z) \tag{6.1.1.13}$$

$$=\frac{\sqrt{-(-1+z)^2}\left(2z\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2\right)-\pi\right)}{-2+2z},\ \text{arccot}(z)=\frac{\pi}{2}-z$$

$${}_2F_1\left(\frac{1}{2},1;\frac{3}{2};-z^2\right),\ \text{arccoth}(z)$$

$$= \frac{\pi \sqrt{-(z-1)^2} + 2\, z\, {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right)(z-1)}{2z-2}, \operatorname{arccsc}(z)$$

$$= \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1}{z^2}\right)}{z}, \operatorname{arccsch}(z) = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -\frac{1}{z^2}\right)}{z}, \operatorname{arcsec}(z)$$

$$= \frac{\pi}{2} - \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1}{z^2}\right)}{z}, \operatorname{arcsech}(z)$$

$$= \frac{\sqrt{-\frac{(z-1)^2}{z^2}}\left(z\pi - 2\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1}{z^2}\right)\right)}{2z-2}, \arcsin(z) = z$$

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right), \operatorname{arcsinh}(z) = z\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2\right), \arctan(z) = z$$

$${}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right), \operatorname{arctanh}(z) = z\, {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right), \cos(z) = {}_0F_1\left(; \frac{1}{2};\right.$$

$$\left. -\frac{z^2}{4}\right), \cosh(z) = {}_0F_1\left(; \frac{1}{2}; \frac{z^2}{4}\right), \cot(z) = \frac{{}_0F_1\left(; \frac{1}{2}; -\frac{z^2}{4}\right)}{z\, {}_0F_1\left(; \frac{3}{2}; -\frac{z^2}{4}\right)}, \coth(z)$$

$$= \frac{{}_0F_1\left(; \frac{1}{2}; \frac{z^2}{4}\right)}{z\, {}_0F_1\left(; \frac{3}{2}; \frac{z^2}{4}\right)}, \csc(z) = \frac{1}{z\, {}_0F_1\left(; \frac{3}{2}; -\frac{z^2}{4}\right)}, \operatorname{csch}(z)$$

$$= \frac{1}{z\, {}_0F_1\left(; \frac{3}{2}; \frac{z^2}{4}\right)}, \mathrm{e}^z = {}_0F_0(;; z), \ln(z) = (z-1)\, {}_2F_1(1, 1; 2; -z+1),$$

$$\sec(z) = \frac{1}{{}_0F_1\left(;\frac{1}{2};-\frac{z^2}{4}\right)}, \operatorname{sech}(z) = \frac{1}{{}_0F_1\left(;\frac{1}{2};\frac{z^2}{4}\right)}, \sin(z) = z\,{}_0F_1\left(;\frac{3}{2};-\frac{z^2}{4}\right), \sinh(z) = z\,{}_0F_1\left(;\frac{3}{2};\frac{z^2}{4}\right), \tan(z) = \frac{z\,{}_0F_1\left(;\frac{3}{2};-\frac{z^2}{4}\right)}{{}_0F_1\left(;\frac{1}{2};-\frac{z^2}{4}\right)}, \tanh(z) = \frac{z\,{}_0F_1\left(;\frac{3}{2};\frac{z^2}{4}\right)}{{}_0F_1\left(;\frac{1}{2};\frac{z^2}{4}\right)}\Bigg]$$

All the 'specializations' of the arcsin function

> *FunctionAdvisor*(*specialize*, arcsin)

$$\left[\arcsin(z) = \frac{z\,HC\left(0, \frac{1}{2}, 0, 0, \frac{1}{4}, \frac{z^2}{z^2-1}\right)}{\sqrt{-z^2+1}}, \textit{with no restrictions on } (z)\right], \quad \textbf{(6.1.1.14)}$$

$$\left[\arcsin(z) = z\,HG\left(0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, z^2\right), \textit{with no restrictions on } (z)\right],$$

$$\left[\arcsin(z) = \frac{\pi}{2} + \frac{am^{-1}(\operatorname{arcsec}(z)|1)\ (z-1)}{\sqrt{-(z-1)^2}}, \textbf{ And } \Re(z) \in (0, \pi)\right],$$

$$\left[\arcsin(z) = \frac{z\,\pi\,P_{-\frac{1}{2}}^{\left(\frac{1}{2}, -\frac{1}{2}\right)}(-2z^2+1)}{2}, \textit{with no restrictions on } (z)\right],$$

$$\left[\arcsin(z) = \frac{z\sqrt{\pi}\ (-2z^2+2)^{1/4}\ P_{-\frac{1}{2}}^{-\frac{1}{2}}(-2z^2+1)}{2\,(-2z^2)^{1/4}}, \textit{with no restrictions on } (z)\right], \left[\arcsin(z) = \frac{z\,G_{2,2}^{1,2}\left(-z^2 \middle| \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 0, -\frac{1}{2} \end{matrix}\right)}{2\sqrt{\pi}},\right.$$

with no restrictions on (z)], $\left[\arcsin(z) = \frac{\pi}{2} - \arccos(z),\right.$

with no restrictions on (z)], $\left[\arcsin(z) = \frac{\pi}{2} + \frac{\operatorname{arccosh}(z)\,(z-1)}{\sqrt{-(z-1)^2}},\right.$

with no restrictions on (z)], $\left[\arcsin(z) = \pi - 2\operatorname{arccot}\left(\frac{z}{1+\sqrt{-z^2+1}}\right),\right.$

with no restrictions on (z)], $\left[\arcsin(z)\right.$

$$= \frac{1}{\mathrm{I}\,z+\sqrt{-z^2+1}+1}\Bigg(\Big(2\,\mathrm{I}\sqrt{-z^2+1}+2\,\mathrm{I}$$

$$-2\,z\Big)\operatorname{arccoth}\left(\frac{-\mathrm{I}\,z}{1+\sqrt{-z^2+1}}\right)+\mathrm{I}\,\Big(1$$

$$+\sqrt{-z^2+1}\Big)\,\pi\sqrt{-\left(\frac{\mathrm{I}\,z}{1+\sqrt{-z^2+1}}+1\right)^2}\Bigg),$$

with no restrictions on (z)], $\left[\arcsin(z) = \operatorname{arccsc}\left(\frac{1}{z}\right),\right.$

with no restrictions on (z)], $\left[\arcsin(z) = \mathrm{I}\operatorname{arccsch}\left(\frac{\mathrm{I}}{z}\right),\right.$

with no restrictions on (z)], $\left[\arcsin(z) = \frac{\pi}{2} - \operatorname{arcsec}\left(\frac{1}{z}\right),\right.$

with no restrictions on (z)], $\left[\arcsin(z) = \frac{\pi}{2} + \frac{\operatorname{arcsech}\left(\frac{1}{z}\right)(-1+z)}{\sqrt{-\left(\frac{1}{z}-1\right)^2 z^2}},\right.$

with no restrictions on (z)], $[\arcsin(z) = -\mathrm{I}\operatorname{arcsinh}(\mathrm{I}\,z),$

with no restrictions on (z)], $\left[\arcsin(z) = 2\arctan\left(\frac{z}{1+\sqrt{-z^2+1}}\right),\right.$

with no restrictions on (z)], [$\arcsin(z) = -2\,\mathrm{I}\,\operatorname{arctanh}\left(\frac{\mathrm{I}\,z}{1+\sqrt{-z^2+1}}\right)$,

with no restrictions on (z)], [$\arcsin(z) = z\;{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right)$,

with no restrictions on (z)], [$\arcsin(z) = -\mathrm{I}\ln\left(\mathrm{I}\,z + \sqrt{-z^2+1}\right)$,

with no restrictions on (z)]

General description

- The FunctionAdvisor command provides information on the following topics.

analytic_extension	asymptotic_expansion	branch_cuts	branch_points
calling_sequence	class_members	classify_function	DE
definition	describe	differentiation_rule	display
function_classes	identities	integral_form	known_functions
relate	series	singularities	special_values
specialize	sum_form	synonyms	topics

The FunctionAdvisor command provides information on the following mathematical functions.

abs	AiryAi	AiryBi	AngerJ
arccos	arccosh	arccot	arccoth
arccsc	arccsch	arcsec	arcsech
arcsin	arcsinh	arctan	arctanh
argument	BellB	bernoulli	BesselI
BesselJ	BesselK	BesselY	Beta

binomial	ChebyshevT	ChebyshevU	Chi
Ci	CompleteBellB	cos	cosh
cot	coth	CoulombF	csc
csch	csgn	CylinderD	CylinderU
CylinderV	dawson	dilog	Dirac
doublefactorial	Ei	EllipticCE	EllipticCK
EllipticCPi	EllipticE	EllipticF	EllipticK
EllipticModulus	EllipticNome	EllipticPi	erf
erfc	erfi	euler	exp
factorial	FresnelC	Fresnelf	Fresnelg
FresnelS	GAMMA	GaussAGM	GegenbauerC
HankelH1	HankelH2	harmonic	Heaviside
HermiteH	HeunB	HeunBPrime	HeunC
HeunCPrime	HeunD	HeunDPrime	HeunG
HeunGPrime	HeunT	HeunTPrime	hypergeom
Hypergeom	Im	IncompleteBellB	InverseJacobiAM
InverseJacobiCD	InverseJacobiCN	InverseJacobiCS	InverseJacobiDC
InverseJacobiDN	InverseJacobiDS	InverseJacobiNC	InverseJacobiND
InverseJacobiNS	InverseJacobiSC	InverseJacobiSD	InverseJacobiSN
JacobiAM	JacobiCD	JacobiCN	JacobiCS
JacobiDC	JacobiDN	JacobiDS	JacobiNC

JacobiND	JacobiNS	JacobiP	JacobiSC
JacobiSD	JacobiSN	JacobiTheta1	JacobiTheta2
JacobiTheta3	JacobiTheta4	JacobiZeta	KelvinBei
KelvinBer	KelvinHei	KelvinHer	KelvinKei
KelvinKer	KummerM	KummerU	LaguerreL
LambertW	LegendreP	LegendreQ	LerchPhi
Li	ln	lnGAMMA	log
LommelS1	LommelS2	MathieuA	MathieuB
MathieuC	MathieuCE	MathieuCEPrime	MathieuCPrime
MathieuExponent	MathieuFloquet	MathieuFloquetPrime	MathieuS
MathieuSE	MathieuSEPrime	MathieuSPrime	MeijerG
pochhammer	polylog	Psi	Re
sec	sech	Shi	Si
signum	sin	sinh	SphericalY
Ssi	Stirling1	Stirling2	StruveH
StruveL	tan	tanh	WeberE
WeierstrassP	WeierstrassPPrime	WeierstrassSigma	WeierstrassZeta
WhittakerM	WhittakerW	Wrightomega	Zeta

Like the conversion facility for mathematical functions, the FunctionAdvisor command also works with the concept of function classes and considers assumptions on the function parameters, if any. The FunctionAdvisor command provides information on the following function classes.

`0F1`	`1F1`	`2F1`	Airy
arctrig	arctrigh	Bessel_related	Chebyshev
Cylinder	Ei_related	elementary	Elliptic_doubly_periodic
Elliptic_related	erf_related	GAMMA_related	Hankel
hypergeometric	Jacobi_related	Kelvin	Kummer
Legendre	Lommel	orthogonal_polynomials	Other
Psi_related	Struve_related	trig	trigh
Weierstrass_related	Whittaker	Zeta_related	

- The FunctionAdvisor command can be considered to be between a help and a computational special function facility. Due to the wide range of information this command can handle and in order to facilitate its use, it includes two distinctive features:
  - If you call the FunctionAdvisor command without arguments, it returns information that you can follow until the appropriate information displays.
  - If you call the FunctionAdvisor command with a topic or function misspelled, but a match exists, it returns the information with a warning message.

You do not have to remember the exact Maple name of each mathematical function or the FunctionAdvisor topic. However, to avoid these messages and all FunctionAdvisor verbosity, specify the optional argument quiet when calling the FunctionAdvisor command from another routine.

References

- Cheb-Terrab, E.S. "The function advisor project: A Computer Algebra Handbook of Special Functions", **Proceedings of the Maple Summer Workshop**, University of Waterloo, Ontario, Canada, 2002.

Differential equation representation for generic nonlinear algebraic expressions - their use

"*Compute differential polynomial forms for a given system of non-polynomial, possibly differential, equations - their use*"

The Differential Equations representing arbitrary algebraic expresssions

> $restart; with(PDEtools):$

For ease of reading, these functions are declared to be displayed in a compact way. Also, derivatives are displayed as indexed objects.

> $declare(f(x), g(x, y), \_F1(x, y), \_F2(x, y), \_F3(x, y), prime = x)$

f(x) will now be displayed as f

g(x, y) will now be displayed as g

_F1(x, y) will now be displayed as _F1

_F2(x, y) will now be displayed as _F2

_F3(x, y) will now be displayed as _F3

derivatives with respect to x of functions of one variable will now be displayed with ' **(6.2.1.1)**

Consider the following non-polynomial expression.

> $f(x) = \tan(x)$

$$f = \tan(x) \quad \textbf{(6.2.1.2)}$$

> $dpolyform(\textbf{(6.2.1.2)}, no\_Fn)$

$$[f' = f^2 + 1] \text{ \&where } [f' \neq 0] \quad \textbf{(6.2.1.3)}$$

> $g(x, y) = \tan\left(2x - y^{\frac{1}{2}}\right)$

$$g = \tan\left(2x - \sqrt{y}\right) \quad \textbf{(6.2.1.4)}$$

A differential polynomial system (DPS) is given by:

> $dpolyform(\textbf{(6.2.1.4)}, no\_Fn)$

$$\left[g_x = 2g^2 + 2, {g_y}^2 = \frac{g^4}{4y} + \frac{g^2}{2y} + \frac{1}{4y}\right] \text{ \&where } \left[g_y \neq 0, -g^2 - 1 \neq 0\right] \quad \textbf{(6.2.1.5)}$$

> $pdetest(\textbf{(6.2.1.4)}, \textbf{(6.2.1.5)})$

$$[0, 0] \quad \textbf{(6.2.1.6)}$$

The dpolyform routine can handle systems of equations containing arbitrary compositions of most functions known to Maple, including fractional or abstract powers.

Deriving knowledge: ODE solving methods

A useful side effect of this is that, by asking for the DP form of a special function for some ***very generic arguments***, one receives the polynomial differential equation it satisfies. For example,

> $y(x) = {}_2F_1(a, b; c; (\alpha x + \beta))$

$$y(x) = {}_2F_1(a, b; c; \alpha x + \beta) \qquad (6.2.2.1)$$

> $dpolyform((6.2.2.1), [y, f], no\_Fn)$

$$\left[y'' = -\frac{\alpha (x (a+b+1) \alpha + (a+b+1) \beta - c) y'}{(\alpha x + \beta)(\alpha x + \beta - 1)} - \frac{\alpha^2 a b y(x)}{(\alpha x + \beta)(\alpha x + \beta - 1)}\right] \&\text{where } [y(x) \neq 0] \qquad (6.2.2.2)$$

So, if we can recast a given equation in the form above as to tell the values of $\{a, \alpha, b, \beta, \delta, \gamma\}$, we directly know its answer:

> $dsolve((6.2.2.2))$

$$\left\{y(x) = \_C1\ {}_2F_1(a, b; c; \alpha x + \beta) + \_C2\ (\alpha x + \beta)^{1-c}\ {}_2F_1(a + 1 - c, b + 1 - c; 2 - c; \alpha x + \beta)\right\} \qquad (6.2.2.3)$$

The same for the more general case

> $y(x) = {}_2F_1\left(a, b; c; \frac{(\alpha x + \beta)}{(\gamma x + \delta)}\right)$

$$y(x) = {}_2F_1\left(a, b; c; \frac{\alpha x + \beta}{\gamma x + \delta}\right) \qquad (6.2.2.4)$$

> $dpolyform((6.2.2.4), y(x), no\_Fn)$

$$\left[y'' = \left(\left(-2 \alpha \gamma (\alpha - \gamma) x^2 + \left(-\beta (c-2) \gamma^2 + \alpha ((c+2) \delta + \beta (a + b - 3)) \gamma - \alpha^2 \delta (a + b + 1)\right) x + \beta ((2-c) \delta + \beta (a+b-1)) \gamma - (-c \delta + (a+b+1) \beta) \alpha \delta\right) y'\right) \Big/ \left((\gamma x + \delta)((\alpha - \gamma) x + \beta - \delta)(\alpha x + \beta)\right) - \frac{a b (-\alpha \delta + \beta \gamma)^2 y(x)}{((\alpha - \gamma) x + \beta - \delta)(\gamma x + \delta)^2 (\alpha x + \beta)}\right] \&\text{where } [y(x) \neq 0] \qquad (6.2.2.5)$$

Extending the mathematical language to include the inverse functions

The dpolyform command can handle expressions involving more abstract representations as DESol, RootOf or @@. For example, consider the inverse of the dawson function.

> $y(x) = \mathrm{dawson}^{(-1)}(x)$

$$y(x) = \mathrm{dawson}^{(-1)}(x) \quad \textbf{(6.2.3.1)}$$

The differential polynomial representation of this object is an Abel equation of the 2nd kind which happens to admit symmetries separable by product.

> *dpolyform*((**6.2.3.1**), *no_Fn*)

$$\left[y' = -\frac{1}{2\,y(x)\,x - 1}\right] \&\text{where } [y(x) \neq 0] \quad \textbf{(6.2.3.2)}$$

> *dsolve*(*op*(1, (**6.2.3.2**)))

$$\left[\left\{\_C1 + \frac{1}{2\,\mathrm{I}\,e^{y(x)^2}\,x - \mathrm{erf}(\mathrm{I}\,y(x))\sqrt{\pi}} = 0\right\}\right] \quad \textbf{(6.2.3.3)}$$

> *DEtools*[*odeadvisor*](*op*([1, 1], (**6.2.3.2**)), *help*)

[[_1st_order, _with_symmetry_[F(x)*G(y),0]], [_Abel, 2nd type, class C]] **(6.2.3.4)**

Solving non-polynomial algebraic equations by solving polynomial differential equations

Finally, it is possible to use dpolyform to solve algebraic (nondifferential) systems as well. As an example of this, consider the following non-polynomial, nondifferential system of equations for the unknowns $y(t), z(t)$.

> *declare*((*y*, *z*)(*t*), *prime* = *t*);

y(t) will now be displayed as y

z(t) will now be displayed as z

derivatives with respect to t of functions of one variable will now be displayed with ' **(6.2.4.1)**

> $sys := \left[t - \tan(y(t) + z(t) - \ln(y(t))) = 0, y(t) - e^{-y(t) + z(t) + \arctan(t)} = 0\right]$

$$sys := \left[t + \tan(-y - z + \ln(y)) = 0, y - e^{-y + z + \arctan(t)} = 0\right] \quad \textbf{(6.2.4.2)}$$

Due to the non-trivial *non-polynomial* expressions involved, the Maple solve command in previous releases failed to solve this problem for $y(t)$ and $z(t)$. By using dpolyform you can solve *sys* as follows. First compute a differential polynomial form for *sys*.

> *dpolyform*(*sys*, *no_Fn*)

$$\left[y' = \frac{1}{t^2 + 1}, z' = \frac{1}{y\,(t^2 + 1)}\right] \&\text{where } [y + 1 \neq 0, y \neq 0] \quad \textbf{(6.2.4.3)}$$

> *DP_sys* := [*op*(*map*(*op*, (**6.2.4.3**)))]

$$DP\_sys := \left[y' = \frac{1}{t^2 + 1}, z' = \frac{1}{y\,(t^2 + 1)}, y + 1 \neq 0, y \neq 0\right] \quad \textbf{(6.2.4.4)}$$

Second, solve *DP_sys* using dsolve

> $sol\_DP\_sys := dsolve(DP\_sys, explicit)$

$$sol\_DP\_sys := \{y = \arctan(t) + \_C2, z = \ln(\arctan(t) + \_C2) + \_C1\} \tag{6.2.4.5}$$

This solution to *DP_sys* includes the solution to sys for some particular values of the integration constants $\{\_C1, \_C2\}$ involved. To determine $\{\_C1, \_C2\}$, substitute the above into sys.

> $sys\_C := eval(sys, sol\_DP\_sys)$

$$sys\_C := \left[t - \tan(\arctan(t) + \_C2 + \_C1) = 0, \arctan(t) + \_C2 - e^{-\_C2 + \ln(\arctan(t) + \_C2) + \_C1} = 0\right] \tag{6.2.4.6}$$

The next step is to solve *sys_C* for $\{\_C1, \_C2\}$. We must expand *sys_C* in a series. It is sufficient to find a few terms from which you can get a solution. For computational reasons, use three intermediate steps.

> $z1 := map(lhs, sys\_C)$

$$z1 := \left[t - \tan(\arctan(t) + \_C2 + \_C1), \arctan(t) + \_C2 - e^{-\_C2 + \ln(\arctan(t) + \_C2) + \_C1}\right] \tag{6.2.4.7}$$

> $z2 := map(series, z1, t, 1)$

$$z2 := \left[-\tan(\_C2 + \_C1) + O(t), \_C2 - e^{-\_C2 + \ln(\_C2) + \_C1} + O(t)\right] \tag{6.2.4.8}$$

> $z3 := simplify(map(convert, z2, polynom))$

$$z3 := \left[-\tan(\_C2 + \_C1), \_C2 - \_C2\, e^{-\_C2 + \_C1}\right] \tag{6.2.4.9}$$

Now solve for $\{\_C1, \_C2\}$:

> $\_EnvAllSolutions := true$

$$\_EnvAllSolutions := true \tag{6.2.4.10}$$

> $sol\_C := solve(\{op(z3)\}, \{\_C1, \_C2\})$

$$sol\_C := \{\_C1 = \pi\,\_Z1, \_C2 = 0\}, \left\{\_C1 = \frac{\pi\,\_Z2}{2} + I\,\pi\,\_Z3, \_C2 = \frac{\pi\,\_Z2}{2} - I\,\pi\,\_Z3\right\} \tag{6.2.4.11}$$

where, by convention, *_Z1~* is an integer (see solve). Finally, the solution above leads to the solution for the nondifferential *sys*, by evaluating sol_DP_sys at these values of the integration constants. To see this, take for instance the solution containing the imaginary unit

> $select(has, [sol\_C], I)_1$

$$\left\{\_C1 = \frac{\pi\,\_Z2}{2} + I\,\pi\,\_Z3, \_C2 = \frac{\pi\,\_Z2}{2} - I\,\pi\,\_Z3\right\} \tag{6.2.4.12}$$

> $sol\_sys := eval(sol\_DP\_sys, \textbf{(6.2.4.12)})$

(6.2.4.13)

$$sol\_sys := \left\{ y = \arctan(t) + \frac{\pi\_Z2}{2} - I\,\pi\,\_Z3,\ z = \ln\left(\arctan(t) + \frac{\pi\_Z2}{2} - I\,\pi\,\_Z3\right) + \frac{\pi\_Z2}{2} + I\,\pi\,\_Z3 \right\} \quad \textbf{(6.2.4.13)}$$

This solution can be verified by substituting into *sys*.

> *sys*

$$[t + \tan(-y - z + \ln(y)) = 0,\ y - e^{-y + z + \arctan(t)} = 0] \quad \textbf{(6.2.4.14)}$$

> *simplify*(*expand*(*eval*(*sys*, *sol_sys*)))

$$[0 = 0,\ 0 = 0] \quad \textbf{(6.2.4.15)}$$

References

- Cheb-Terrab, E.S. "Special functions in Computer Algebra, " **Special Functions in the Digital Age (IMA 2002) Minneapolis, USA.**

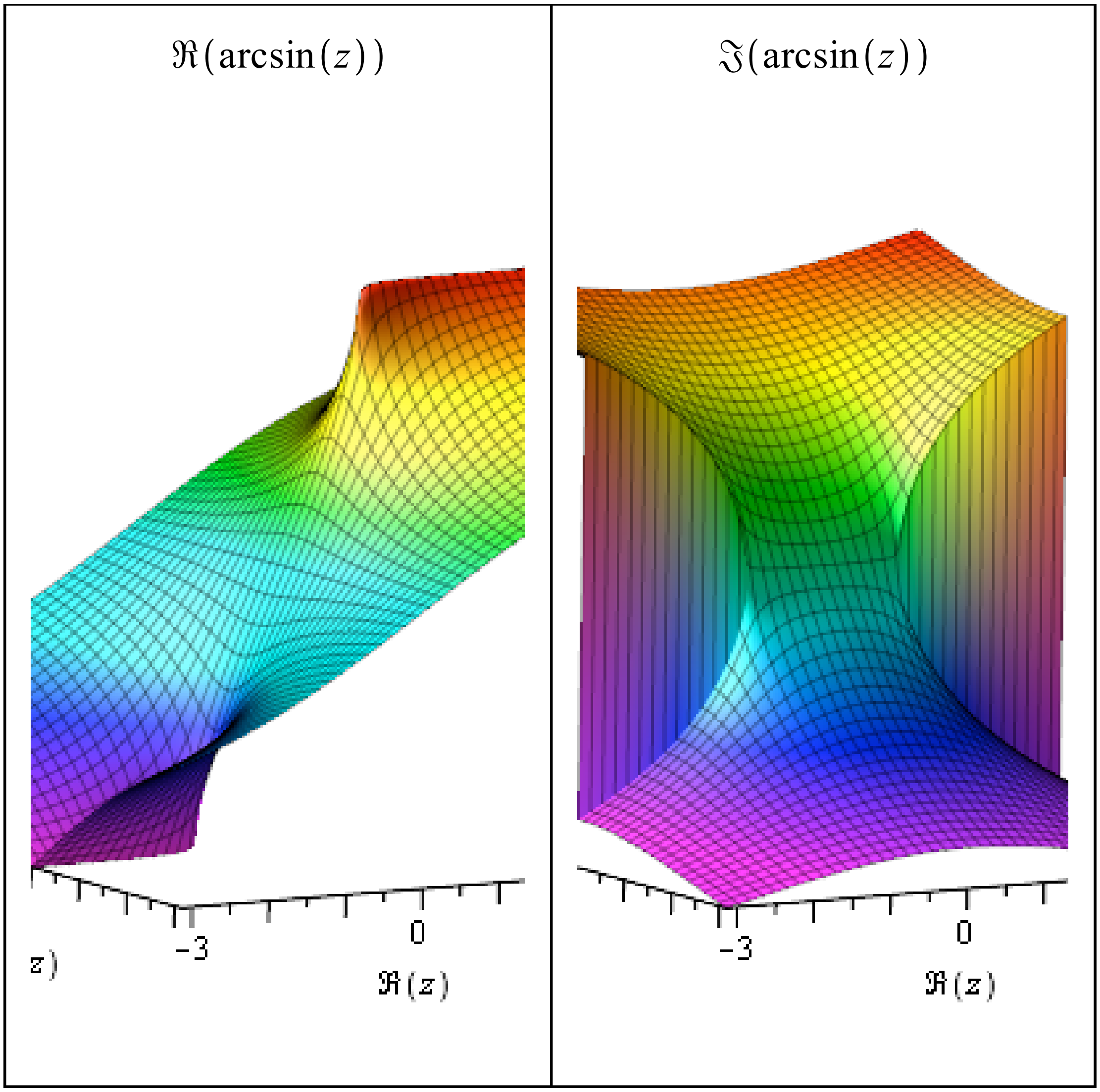
- Cheb-Terrab, E.S. "ODEs, PDEs and Special Functions", presentation, University of Cantabria, Spain, June 2013.

Branch Cuts of algebraic expressions

"Algebraically compute, and visualize, the branch cuts of a given arbitrary mathematical expression"

Examples

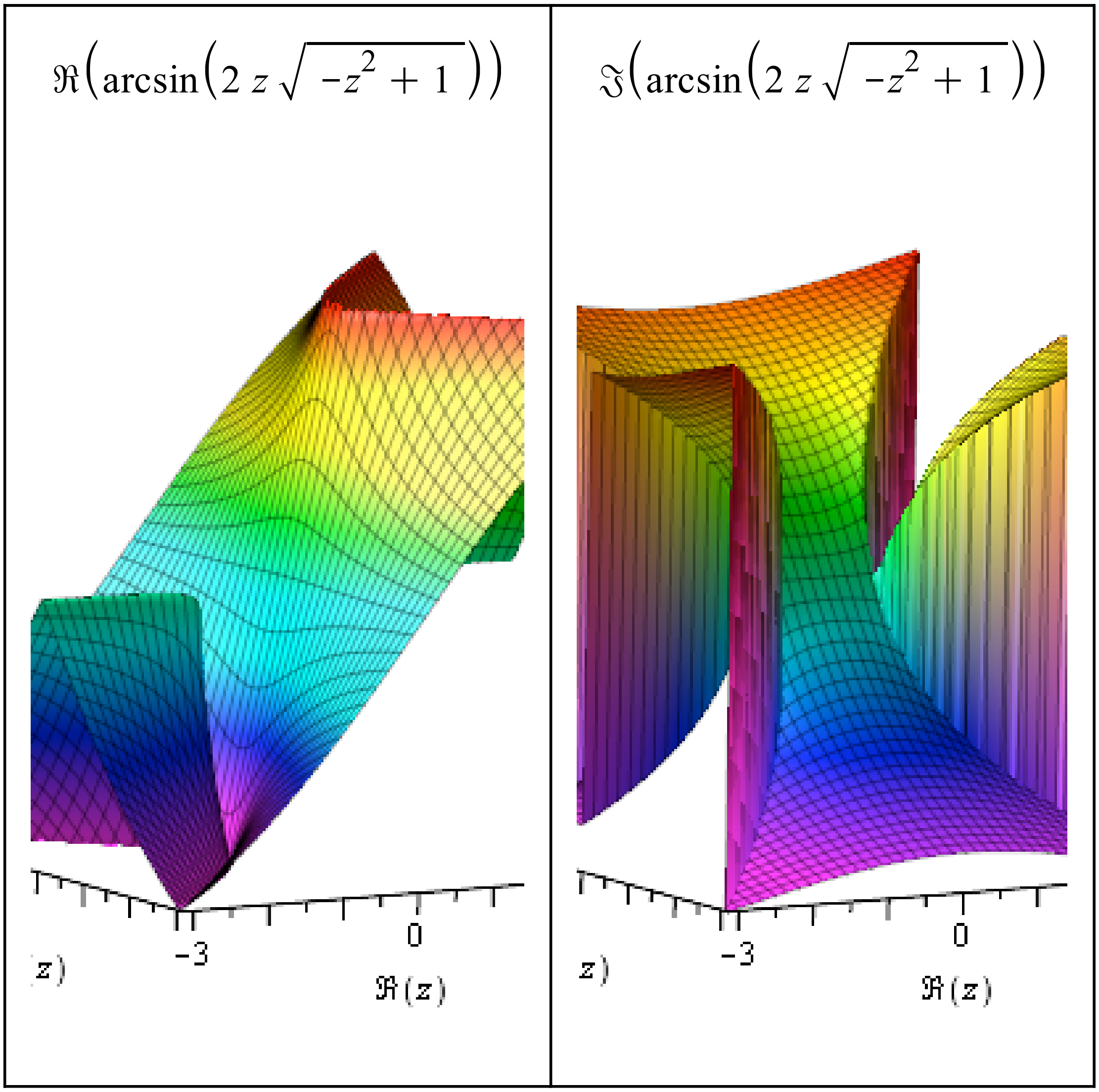
> *FunctionAdvisor*(*branch_cuts*, arcsin(*z*), *plot* = 3 D)



$[\arcsin(z), 1 \leq z, z \leq -1]$ **(6.3.1.1)**

What about the cuts of an algebraic expression?

> $\mathit{FunctionAdvisor}\left(\mathit{branch\_cuts}, \arcsin\left(2\,z\,\sqrt{1-z^2}\right), \mathit{plot} = 3\,\mathrm{D}\right)$



$$\left[\arcsin\left(2\,z\,\sqrt{-z^2+1}\right), 1 < z, z < -1, \Re(z) = -\frac{\sqrt{4\,\Im(z)^2+2}}{2}, \Re(z) = \frac{\sqrt{4\,\Im(z)^2+2}}{2}\right] \quad \textbf{(6.3.1.2)}$$

A classic example in the theory of branch cut calculation is that of Kahan's teardrop:

> $KTD := 2\,\mathrm{arccosh}\left(\frac{(3+(2\,z))}{3}\right) - 2\,\mathrm{arccosh}\left(\frac{(12+(5\,z))}{(3\,(z+4))}\right) - 2\,\mathrm{arccosh}\left(2\,(z+3)\sqrt{\frac{(z+3)}{(27\,(z+4))}}\right):$

> $FunctionAdvisor(branch\_cuts, KTD, plot = 3., shift\_range = -3, scale\_range = 2, orientation = [-124, 20, 14])$

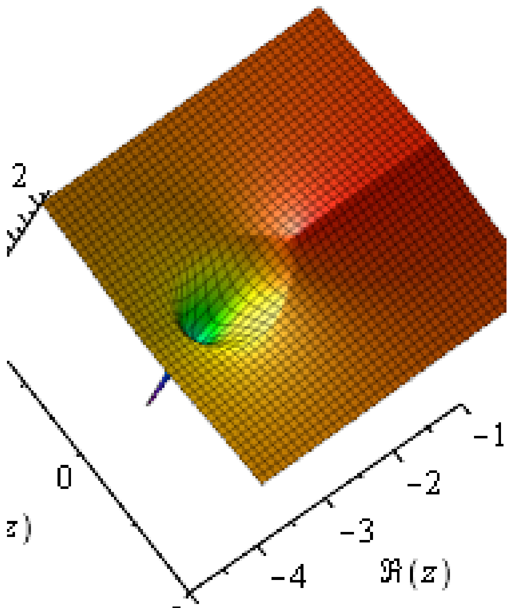
<table>
<tr>
<td>

$\Re\left(2\,\mathrm{arccosh}\left(1+\frac{2}{3}\,z\right)\right.$

$-\,2\,\mathrm{arccosh}(1$

$/(3\,z+12)(12+5\,z))$

$-\,2\,\mathrm{arccosh}\Big(2\,(z$



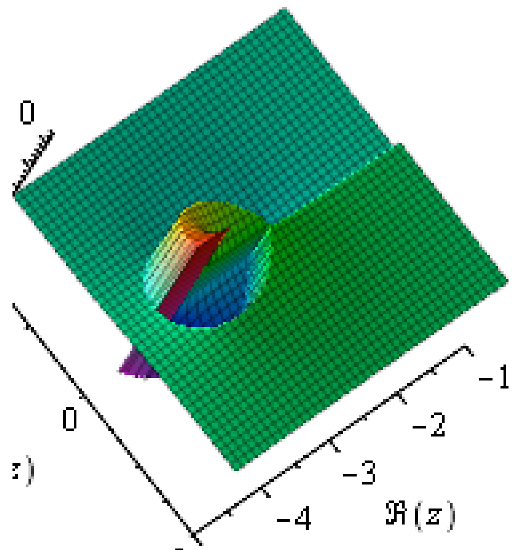
</td>
<td>

$\Im\left(2\,\mathrm{arccosh}\left(1+\frac{2}{3}\,z\right)\right.$

$-\,2\,\mathrm{arccosh}(1$

$/(3\,z+12)(12+5\,z))$

$-\,2\,\mathrm{arccosh}\Big(2\,(z$



</td>
</tr>
</table>

$$\left[2\,\mathrm{arccosh}\left(1+\frac{2\,z}{3}\right)-2\,\mathrm{arccosh}\left(\frac{12+5\,z}{3\,z+12}\right)-2\,\mathrm{arccosh}\Bigg(2\,(z+3)\sqrt{\frac{z+3}{27\,z+108}}\Bigg),-4<z\leq 0,-\frac{9}{2}<z<-4,\Im(z)=\frac{\sqrt{-4\,\Re(z)^2-28\,\Re(z)-45}\,(\Re(z)+3)}{2\,\Re(z)+5}\ \textbf{And}\ \Re(z)\leq 0\ \textbf{And}\ -\frac{9}{4}<\Re(z),\Im(z)=\frac{\sqrt{-4\,\Re(z)^2-28\,\Re(z)-45}\,(\Re(z)+3)}{2\,\Re(z)+5}\ \textbf{And}\ \Re(z)\leq -3\ \textbf{And}\ -\frac{9}{2}<\Re(z),\Im(z)=-\frac{\sqrt{-4\,\Re(z)^2-28\,\Re(z)-45}\,(\Re(z)+3)}{2\,\Re(z)+5}\ \textbf{And}\ \Re(z)\leq 0\ \textbf{And}\ -\frac{9}{4}\right. \tag{6.3.1.3}$$

$$< \Re(z), \Im(z) = \frac{\sqrt{-4\,\Re(z)^2 - 28\,\Re(z) - 45}\,(\Re(z)+3)}{2\,\Re(z)+5}\ \textbf{And}\ -\frac{5}{2}$$

$$< \Re(z)\ \textbf{And}\ \Re(z) < -\frac{9}{4}, \Im(z) =$$

$$-\frac{\sqrt{-4\,\Re(z)^2 - 28\,\Re(z) - 45}\,(\Re(z)+3)}{2\,\Re(z)+5}\ \textbf{And}\ \Re(z) \leq -3\ \textbf{And}\ -\frac{9}{2}$$

$$< \Re(z), \Im(z) = -\frac{\sqrt{-4\,\Re(z)^2 - 28\,\Re(z) - 45}\,(\Re(z)+3)}{2\,\Re(z)+5}\ \textbf{And}\ -\frac{5}{2}$$

$$< \Re(z)\ \textbf{And}\ \Re(z) < -\frac{9}{4}\Bigg]$$

References

- Cheb-Terrab, E.S., England M., Bradford R., Davenport J.H., Wilson, D. "Branch cuts of algebraic expressions", ACM **Communications in Computer Algebra 48:1 pp. 24-27, ACM, 2014**.

- Cheb-Terrab, E.S. "The function wizard project: A Computer Algebra Handbook of Special Functions". **Proceedings of the Maple Summer Workshop**, University of Waterloo, Ontario, Canada, 2002.

Algebraic expresssions in terms of specified functions

"A conversion network for mathematical expressions, allowing to rewrite algebraic expressions in terms of different functions in flexible ways"

Examples

> *restart*;

Start with the error function

> erf(*z*)

$$\operatorname{erf}(z) \qquad \textbf{(6.4.1.1)}$$

> *convert*(**(6.4.1.1)**, HermiteH)

$$-\frac{2\,H_{-1}(z)}{\sqrt{\pi}\,e^{z^2}} + 1 \qquad \textbf{(6.4.1.2)}$$

> *convert*(**(6.4.1.2)**, KummerU)

$$-\frac{z\,U\left(1,\frac{3}{2},z^2\right)\sqrt{z^2}-\sqrt{\pi}\,e^{z^2}\left(z-\sqrt{z^2}\right)}{\sqrt{z^2}\,\sqrt{\pi}\,e^{z^2}}+1 \qquad \textbf{(6.4.1.3)}$$

> *normal*(*convert*(**(6.4.1.3)**, WhittakerW))

$$\frac{z\left(\sqrt{\pi}\,e^{z^2}\left(z^2\right)^{1/4}-W_{-\frac{1}{4},\frac{1}{4}}\left(z^2\right)e^{\frac{z^2}{2}}\right)}{\sqrt{\pi}\,e^{z^2}\left(z^2\right)^{3/4}} \qquad \textbf{(6.4.1.4)}$$

> *convert*(**(6.4.1.4)**, hypergeom)

$$\frac{z\left(\sqrt{\pi}\,e^{z^2}\left(z^2\right)^{1/4}-\left(z^2\right)^{1/4}\sqrt{\pi}\,{}_0F_0\left(;;z^2\right)+2\left(z^2\right)^{3/4}\,{}_1F_1\left(1;\frac{3}{2};z^2\right)\right)}{\sqrt{\pi}\,e^{z^2}\left(z^2\right)^{3/4}} \qquad \textbf{(6.4.1.5)}$$

> *simplify*(**(6.4.1.5)**)

$$\mathrm{erf}(z) \qquad \textbf{(6.4.1.6)}$$

Most special functions satisfy identities relating contiguous values of their parameters. These identities can also be computed as "conversions", for instance,

> $H_a(z)$

$$H_a(z) \qquad \textbf{(6.4.1.7)}$$

> **(6.4.1.7)** = *convert*(**(6.4.1.7)**, HermiteH, "raise a")

$$H_a(z)=\frac{2\,z\,H_{a+1}(z)-H_{a+2}(z)}{2\,a+2} \qquad \textbf{(6.4.1.8)}$$

Many "rule conversions" can be requested at once and performed in a specified order; for instance, consider the following expression:

> $ee := \frac{1}{8}\,U\left(\frac{3}{2},\frac{1}{2},z^2\right)-\frac{1}{8}\,U\left(1,\frac{3}{2},z^2\right)\sqrt{z^2}-\frac{1}{4}\,U\left(1,\frac{3}{2},z^2\right)z^2\sqrt{z^2}+\left(\frac{1}{4}\sqrt{z^2}\right)$

$$ee := \frac{U\left(\frac{3}{2},\frac{1}{2},z^2\right)}{8}-\frac{U\left(1,\frac{3}{2},z^2\right)\sqrt{z^2}}{8}-\frac{U\left(1,\frac{3}{2},z^2\right)z^2\sqrt{z^2}}{4}+\frac{\sqrt{z^2}}{4} \qquad \textbf{(6.4.1.9)}$$

By converting it to 'KummerU' and applying rules for normalizing the first and second indices, here respectively called *a* and *b*, and then mixing them, you obtain varied representations for the same expression.

> *convert*(*ee*, KummerU, "normalize a", "normalize b", "mix a and b")

$$\frac{\sqrt{z^2}\, U\left(2, \frac{3}{2}, z^2\right)}{8} - \frac{U\left(\frac{1}{2}, \frac{1}{2}, z^2\right)}{8} - \frac{U\left(\frac{1}{2}, \frac{1}{2}, z^2\right) z^2}{4} + \frac{\sqrt{z^2}}{4} \qquad \textbf{(6.4.1.10)}$$

> *convert*(**(6.4.1.10)**, KummerU, "normalize a", "normalize b", "mix a and b")

$$\frac{\left(2 z^2 + 1\right) U\left(\frac{1}{2}, \frac{1}{2}, z^2\right)}{8} - \frac{U\left(1, \frac{3}{2}, z^2\right) \sqrt{z^2}}{8} - \frac{U\left(1, \frac{3}{2}, z^2\right) z^2 \sqrt{z^2}}{4} \qquad \textbf{(6.4.1.11)}$$

A further manipulation actually shows the expression was always equal to zero :)

> *convert*(**(6.4.1.11)**, KummerU, "normalize a", "normalize b", "mix a and b")

$$\frac{\left(2 z^2 + 1\right) U\left(\frac{1}{2}, \frac{1}{2}, z^2\right)}{8} - \frac{U\left(\frac{1}{2}, \frac{1}{2}, z^2\right)}{8} - \frac{U\left(\frac{1}{2}, \frac{1}{2}, z^2\right) z^2}{4} \qquad \textbf{(6.4.1.12)}$$

> *normal*(**(6.4.1.12)**)

$$0 \qquad \textbf{(6.4.1.13)}$$

General description

• The functions to which you can convert - the second argument could be one of these - are:

AiryAi	AiryBi	arccos	arccosh
arccot	arccoth	arccsc	arccsch
arcsec	arcsech	arcsin	arcsinh
arctan	arctanh	BellB	BesselI
BesselJ	BesselK	BesselY	ChebyshevT
ChebyshevU	cos	cosh	cot
coth	CoulombF	csc	csch
CylinderD	CylinderU	CylinderV	dilog
Ei	EllipticE	EllipticK	erf
exp	GAMMA	GegenbauerC	HankelH1
HankelH2	HermiteH	hypergeom	JacobiP

KummerM	KummerU	LaguerreL	LegendreP
LegendreQ	ln	MeijerG	polylog
Psi	sec	sech	sin
sinh	SphericalY	tan	tanh
WhittakerM	WhittakerW	Wrightomega	

- When a *function class* name is given as second argument the routines attempt to express expr using any of the functions of that class. The function classes understood by convert are:

\`0F1\`	\`1F1\`	\`2F1\`	Airy
arctrig	arctrigh	Bessel_related	Chebyshev
Cylinder	Ei_related	elementary	Elliptic_related
erf_related	GAMMA_related	Hankel	Heun
Kelvin	Kummer	Legendre	trig
trigh	Whittaker		

* (note spaces between words are filled with _)

- Despite the large number of function classes, most of the functions of mathematical physics belong to one of the three hypergeometric classes: 2F1, 1F1, and 0F1 where these three classes also include as particular cases all the elementary functions (trig, hyperbolic trig, their arcs, exp, and ln).

The Optional Arguments

The conversion routines accept an optional extra argument indicating a *rule*; it could be one rule or a sequence of them:

"raise a",	"lower a",	"normalize a",
"raise b",	"lower b",	"normalize b",
"raise c",	"lower c",	"normalize c",
"mix a and b",	"1F1 to 0F1",	"0F1 to 1F1"

"quadratic 1", "quadratic 2", "quadratic 3"

"quadratic 4", "quadratic 5", "quadratic 6"

"2a2b", "raise 1/2", "lower 1/2"

References

- Cheb-Terrab, E.S. "The function advisor project: A Computer Algebra Handbook of Special Functions", **Proceedings of the Maple Summer Workshop**, University of Waterloo, Ontario, Canada, 2002.

Symbolic differentiation of algebraic expressions

"Perform symbolic differentiation by combining different algebraic techniques, including functions of symbolic sequences and Faà di Bruno's formula"

Examples

A power where the exponent is linear in the differentiation variable, a relatively easy problem, can be in a database

> *restart*;

> $(\%diff = diff)\left(f^{\alpha z + \beta}, z\$n\right)$

$$\frac{\partial^n}{\partial z^n}\left(f^{\alpha z + \beta}\right) = \alpha^n f^{\alpha z + \beta} \ln(f)^n \qquad \textbf{(6.5.1.1)}$$

More complicated, consider the k^{th} power of a generic function; the corresponding symbolic derivative can be mapped into a sum of symbolic derivatives of powers of $g(z)$ with lower degree

> $(\%diff = diff)\left(g(\alpha z + \beta)^k, z\$n\right)$ assuming $n < k$

$$\frac{\partial^n}{\partial z^n}\left(g(\alpha z + \beta)^k\right) = k \binom{n-k}{n} \left(\sum_{\_k1=0}^{n} \frac{1}{k - \_k1}\left((-1)^{\_k1} \binom{n}{\_k1} g(\alpha z + \beta)^{k - \_k1} \frac{\partial^n}{\partial z^n}\left(g(\alpha z + \beta)^{\_k1}\right)\right)\right) \qquad \textbf{(6.5.1.2)}$$

For $g = \ln$ this result can be expressed using Stirling numbers of the first kind

> $(\%diff = diff)\left(\ln(\alpha z + \beta)^k, z\$n\right)$

$$\frac{\partial^n}{\partial z^n}\left(\ln(\alpha z+\beta)^k\right)=\frac{\alpha^n\left(\sum_{\_k1=0}^{n}(k-\_k1+1)_{\_k1}\,S_n^{\_k1}\ln(\alpha z+\beta)^{k-\_k1}\right)}{(\alpha z+\beta)^n} \quad \textbf{(6.5.1.3)}$$

The case of the MeijerG function of an *arbitrary number of parameters*

> MeijerG([[a[i] $ i = 1 ..n], [b[i] $ i = n + 1 ..p]], [[b[i] $ i = 1 ..m], [b[i] $ i = m + 1 ..q]], z)

$$G_{p,\,q}^{m,\,n}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_n,b_{n+1},\ldots,b_p\\ b_1,\ldots,b_m,b_{m+1},\ldots,b_q\end{matrix}\right) \quad \textbf{(6.5.1.4)}$$

> diff((6.5.1.4), z$k)

$$G_{1+p,\,q+1}^{m,\,n+1}\left(z\,\middle|\,\begin{matrix}-k,a_1-k,\ldots,a_n-k,b_{n+1}-k,\ldots,b_p-k\\ b_1-k,\ldots,b_m-k,0,b_{m+1}-k,\ldots,b_q-k\end{matrix}\right) \quad \textbf{(6.5.1.5)}$$

Not only the mathematics of this result is correct: the object returned is actually computable (for given concrete values of *n*, *p*, *m* and *q*)

Finally, the "holy grail" of this problem: the n^{th} derivative of a composite function $f(g(z))$ - this always-unreachable implementation of Faa di Bruno formula. Building on top of the basic blocks of knowledge, Faa di Bruno's formula is now a new, higher level, basic block:

> (%diff = diff)(f(g(z)), z$n)

$$\frac{d^n}{dz^n}f(g(z))=\sum_{\_k2=0}^{n}D^{(\_k2)}(f)(g(z))\,\mathrm{IncompleteBellB}\left(n,\_k2,\frac{d}{dz}g(z),\ldots,\frac{d^{n-\_k2+1}}{dz^{n-\_k2+1}}g(z)\right) \quad \textbf{(6.5.1.6)}$$

Note the symbolic sequence of symbolic order derivatives of lower degree, both of of *f* and *g*, also within the arguments of the IncompleteBellB function. This is a ***very*** abstract formula ...

To see this at work, consider, for instance, a case where the n^{th} derivatives of $f(z)$ and $g(z)$ can both be computed, say $f = \sin$, $g = \cos$

> sin(cos(α z + β))

$$\sin(\cos(\alpha z+\beta)) \quad \textbf{(6.5.1.7)}$$

> (%diff = diff)((6.5.1.7), z$n)

$$\frac{d^n}{dz^n}\sin(\cos(\alpha z+\beta))=\sum_{\_k2=0}^{n}\sin\left(\cos(\alpha z+\beta)\right. \quad \textbf{(6.5.1.8)}$$

$$+ \frac{\_k2\,\pi}{2}\Big)\,\mathrm{IncompleteBellB}\Big(n, \_k2, \cos\Big(\alpha z + \beta + \frac{\pi}{2}\Big)\,\alpha, ..., \cos\Big(\alpha z + \beta + \frac{(n - \_k2 + 1)\,\pi}{2}\Big)\,\alpha^{n - \_k2 + 1}\Big)$$

Take for instance $n = 3$

> *eval*(**(6.5.1.8)**, $n = 3$)

$$\frac{d^3}{dz^3}\sin(\cos(\alpha z + \beta)) = \sum_{\_k2=0}^{3} \sin\Big(\cos(\alpha z + \beta) + \frac{\_k2\,\pi}{2}\Big)\,\mathrm{IncompleteBellB}\Big(3, \_k2, \cos\Big(\alpha z + \beta + \frac{\pi}{2}\Big)\,\alpha, ..., \cos\Big(\alpha z + \beta + \frac{(4 - \_k2)\,\pi}{2}\Big)\,\alpha^{4 - \_k2}\Big) \quad \textbf{(6.5.1.9)}$$

> *value*(**(6.5.1.9)**)

$$\alpha^3 \sin(\alpha z + \beta)\cos(\cos(\alpha z + \beta)) - 3\,\alpha^3 \cos(\alpha z + \beta)\sin(\alpha z + \beta)\sin(\cos(\alpha z + \beta)) + \alpha^3 \sin(\alpha z + \beta)^3 \cos(\cos(\alpha z + \beta)) = \alpha^3 \sin(\alpha z + \beta)\cos(\cos(\alpha z + \beta)) - 3\,\alpha^3 \cos(\alpha z + \beta)\sin(\alpha z + \beta)\sin(\cos(\alpha z + \beta)) + \alpha^3 \sin(\alpha z + \beta)^3 \cos(\cos(\alpha z + \beta)) \quad \textbf{(6.5.1.10)}$$

The two sides of this equation are equal

> *simplify*((*lhs* − *rhs*)(**(6.5.1.10)**))

$$0 \quad \textbf{(6.5.1.11)}$$

References

- Cheb-Terrab, E.S. "The function advisor project: A Computer Algebra Handbook of Special Functions", **Proceedings of the Maple Summer Workshop**, University of Waterloo, Ontario, Canada, 2002.

- Cheb-Terrab, E.S.; von Bülow, Katherina; "Nth order derivatives and Faa di Bruno formula", Mapleprimes blog, August 2015.

Ordinary Differential Equations

"Beyond the concept of a database, classify an arbitrary ODE and suggest solution methods for it"

General description

• Given an ODE, the `odeadvisor` command's main goal is to classify it according to standard text books (see dsolve,references), and to display a help page including related information for solving it .

- First order ODEs

Abel,	Abel2A,	Abel2C,	Bernoulli,	Chini,
Clairaut,	dAlembert,	exact,	fully_exact_linear,	homogeneous,
homogeneousB,	homogeneousC,	homogeneousD,	homogeneousG,	linear,
patterns,	quadrature,	rational,	Riccati,	separable,
sym_implicit				

- Second order ODEs

Bessel,	Duffing,	ellipsoidal,	elliptic,	Emden,
erf,	exact_linear,	exact_nonlinear,	Gegenbauer,	Halm,
Hermite,	Jacobi,	Lagerstrom,	Laguerre,	Lienard,
Liouville,	linear_ODEs,	linear_sym,	missing,	Painleve,
quadrature,	reducible,	sym_Fx,	Titchmarsh,	Van_der_Pol

- High order ODEs

quadrature,	missing,	exact_linear,	exact_nonlinear,	reducible,
linear_ODEs				

- The following links point to help pages containing details of the implementation of specific ODE solving algorithms:

First order Abel ODEs	First order symmetry patterns
Algorithms for first order ODEs	Algorithms for Linear ODEs
Factorization of Linear ODEs	Integrating Factors for Linear ODEs
Algorithm for fully exact Linear ODEs	Integrating Factors for 2nd order Non-Line ODEs
Algorithms for exact solutions for ODEs of all orders	dsolve in Education
Hypergeometric solutions to second order linear	

ODEs

Examples

Kamke's ODE 97

> $restart; with(DEtools) : PDEtools\text{:-}declare(y(x), prime = x)$

y(x) will now be displayed as y

derivatives with respect to x of functions of one variable will now be displayed with ' **(7.2.1)**

> $ODE := x\, y'(x) + \left(a\, y(x)^2\right) - y(x) + \left(b\, x^2\right) = 0$

$$ODE := x\, y' + a\, y^2 - y + b\, x^2 = 0 \quad \textbf{(7.2.2)}$$

> $odeadvisor(ODE)$

[[_homogeneous, class D], _rational, _Riccati] **(7.2.3)**

> $odeadvisor(ODE, help)$

[[_homogeneous, class D], _rational, _Riccati] **(7.2.4)**

References

- Kamke, E. **Differentialgleichungen: Losungsmethoden und Losungen**. New York: Chelsea Publishing Company, 1959.

- Zwillinger, D. **Handbook of Differential Equations**. 2d ed. Orlando, Florida: Academic Press, 1992.

- Cheb-Terrab, E.S.; Duarte, L.G.S.; and da Mota, L.A.C.P. "Computer Algebra Solving of First Order ODEs Using Symmetry Methods." **Computer Physics Communications**. Vol. **101**. (1997): 254.

Exact Solutions to Einstein's equations

$$g_{\mu, \nu}\, \Lambda + G_{\mu, \nu} = 8\, \pi\, T_{\mu, \nu}$$

*"The authors of "Exact solutions toEinstein's equations" reviewed more than **4,000 papers containing solutions to Einstein's equations** in the general relativity literature, organized*

the whole material into chapters according to the physical properties of these solutions. These solutions are key in the area of general relativity and are now all digitized"

The ability to search the database according to the physical properties of the solutions, their classification, or just by parts of keywords (old paradigm) changes the game.

More important, within a computer algebra system *this knowledge becomes alive* (new paradigm).

- The solutions are turned active by a simple call to one command, called the g_ spacetime metric.
- Everything else gets automatically derived and set on the fly (Christoffel symbols , Ricci and Riemann tensors orthonormal and null tetrads, etc.)
- Almost all of the mathematical operations one can perform on these solutions are implemented as commands in the Physics and DifferentialGeometry packages.
- All the mathematics within the Maple library are instantly ready to work with these solutions and derived mathematical objects.

Finally, in the Maple PDEtools package, we have all the mathematical tools to tackle *the equivalence problem around these solutions*.

Examples

Load Physics, set the metric to be *Schwarzschild's* solution (and everything else automatically) in one go

> $restart;\ with(Physics):$

> $g\_[sc]$

Systems of spacetime Coordinates are: $\{X = (r, \theta, \phi, t)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (r, \theta, \phi, t)\}$

The Schwarzschild metric in coordinates $[r, \theta, \phi, t]$

Parameters: $[m]$

$$g_{\mu,\nu} = \begin{bmatrix} \frac{r}{-r+2m} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & \frac{r-2m}{r} \end{bmatrix} \tag{8.1.1}$$

And that is all we do.

All of the tensor components of the general relativity tensors related to this solution got derived automatically, on the fly. For instance this is the definition (in terms of Christoffel symbols) of the Riemann tensor followed by its 16 invariants using the formulas by Carminati and McLenaghan

> $Riemann[definition]$

$$R_{\mu, \nu, \alpha, \beta} = g_{\mu, \lambda} \left(\partial_\alpha\left(\Gamma^{\lambda}{}_{\nu, \beta}\right) - \left(\partial_\beta\left(\Gamma^{\lambda}{}_{\nu, \alpha}\right)\right) + \Gamma^{\lambda}{}_{\kappa, \alpha} \Gamma^{\kappa}{}_{\nu, \beta} - \Gamma^{\lambda}{}_{\kappa, \beta} \Gamma^{\kappa}{}_{\nu, \alpha}\right) \quad \textbf{(8.1.2)}$$

> *Riemann*[*invariants*]

$$r_0 = 0, r_1 = 0, r_2 = 0, r_3 = 0, w_1 = \frac{6\, m^2}{r^6}, w_2 = \frac{6\, m^3}{r^9}, m_1 = 0, m_2 = 0, m_3 = 0, m_4 = 0, m_5 = 0 \quad \textbf{(8.1.3)}$$

The related Weyl scalars in the context of the Newman-Penrose formalism

> *Weyl*[*scalarsdefinition*]

$$\psi_0 = -C^{\mu, \nu, \alpha, \beta}\, l_\mu\, m_\nu\, l_\alpha\, m_\beta, \psi_1 = -C^{\mu, \nu, \alpha, \beta}\, l_\mu\, n_\nu\, l_\alpha\, m_\beta, \psi_2 = -C^{\mu, \nu, \alpha, \beta}\, l_\mu\, m_\nu\, \overline{m}_\alpha\, n_\beta, \psi_3 = -C^{\mu, \nu, \alpha, \beta}\, l_\mu\, n_\nu\, \overline{m}_\alpha\, n_\beta, \psi_4 = -C^{\mu, \nu, \alpha, \beta}\, n_\mu\, \overline{m}_\nu\, n_\alpha\, \overline{m}_\beta \quad \textbf{(8.1.4)}$$

> *Weyl*[*scalars*]

$$\psi_0 = 0, \psi_1 = 0, \psi_2 = -\frac{m}{r^3}, \psi_3 = 0, \psi_4 = 0 \quad \textbf{(8.1.5)}$$

The four Killing vectors defining transformations that leave the spacetime metric solution invariant

> *Define*(*K*, *quiet*) :

> *KillingVectors*(*K*)

$$\left[K^\mu = [0, 0, 0, 1], K^\mu = \left[0, \sin(\phi), \frac{\cos(\phi)}{\tan(\theta)}, 0\right], K^\mu = \left[0, \cos(\phi), -\frac{\sin(\phi)}{\tan(\theta)}, 0\right], K^\mu = [0, 0, 1, 0]\right] \quad \textbf{(8.1.6)}$$

These are the 2x2 matrix components of the Christoffel symbols of the second kind (that describe, in coordinates, the effects of parallel transport in curved surfaces), when the first of its three indices is equal to 1

> *Christoffel*[~1, alpha, beta, *matrix*]

$$\Gamma^{1}{}_{\alpha, \beta} = \begin{bmatrix} \frac{m}{r\,(-r + 2\,m)} & 0 & 0 & 0 \\ 0 & -r + 2\,m & 0 & 0 \\ 0 & 0 & (-r + 2\,m)\sin(\theta)^2 & 0 \\ 0 & 0 & 0 & \frac{-2\,m^2 + m\,r}{r^3} \end{bmatrix} \quad \textbf{(8.1.7)}$$

> *Christoffel*[1, alpha, beta, *matrix*]

$$\Gamma_{1, \alpha, \beta} = \begin{bmatrix} \frac{m}{(-r+2\,m)^2} & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r\sin(\theta)^2 & 0 \\ 0 & 0 & 0 & -\frac{m}{r^2} \end{bmatrix} \tag{8.1.8}$$

This is the tetrad related to the book's metric with number 12.16.1

> *with*(*Tetrads*) :

Setting lowercaselatin letters to represent tetrad indices

Defined as tetrad tensors (*see ?Physics,tetrads*), $e_{a, \mu}$, $\eta_{a, b}$, $\gamma_{a, b, c}$, $\lambda_{a, b, c}$

Defined as spacetime tensors representing the NP null vectors of the tetrad formalism (*see ?Physics,tetrads*), l_μ, n_μ, m_μ, $\overline{m}_\mu$ **(8.1.9)**

> *e*_[ ]

$$e_{a, \mu} = \begin{bmatrix} \frac{-I\sqrt{r}}{\sqrt{-r+2\,m}} & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r\sin(\theta) & 0 \\ 0 & 0 & 0 & \frac{-I\sqrt{-r+2\,m}}{\sqrt{r}} \end{bmatrix} \tag{8.1.10}$$

One can check these things directly; for instance this is the definition of the tetrad, where the right-hand side is the tetrad metric

> *e*_[*definition*]

$$e_{a, \mu}\, e_b^{\ \mu} = \eta_{a, b} \tag{8.1.11}$$

This shows that, for the components given by **(8.1.10)**, the definition holds

> *TensorArray*(**(8.1.11)**)

$$\begin{bmatrix} -1 = -1 & 0 = 0 & 0 = 0 & 0 = 0 \\ 0 = 0 & -1 = -1 & 0 = 0 & 0 = 0 \\ 0 = 0 & 0 = 0 & -1 = -1 & 0 = 0 \\ 0 = 0 & 0 = 0 & 0 = 0 & 1 = 1 \end{bmatrix} \tag{8.1.12}$$

And where are Einstein's equations?

> *Einstein*[mu, nu] = 8 Pi *T*[mu, nu]

$$G_{\mu,\nu} = 8\,\pi\,T_{\mu,\nu} \tag{8.1.13}$$

The expression on the left represents the curvature of spacetime as determined by the metric; the expression on the right represents the matter/energy content of spacetime.

> *Einstein*[*definition*]

$$G_{\mu,\nu} = R_{\mu,\nu} - \frac{g_{\mu,\nu}\,R_{\alpha}{}^{\alpha}}{2} \tag{8.1.14}$$

Substituting in **(8.1.13)**

> *isolate*(*subs*(**(8.1.14)**, **(8.1.13)**), *T*[mu, nu])

$$T_{\mu,\nu} = \frac{R_{\mu,\nu}}{8\,\pi} \tag{8.1.15}$$

> *convert*(**(8.1.15)**, *g_*)

$$\begin{aligned} T_{\mu,\nu} = \frac{1}{8\,\pi}\Bigg(& \frac{\left(\partial_{\alpha}\left(g^{\alpha,\tau}\right)\right)\left(\partial_{\nu}\left(g_{\mu,\tau}\right)+\partial_{\mu}\left(g_{\nu,\tau}\right)-\left(\partial_{\tau}\left(g_{\mu,\nu}\right)\right)\right)}{2} \\ & + \frac{g^{\alpha,\tau}\left(\partial_{\alpha}\left(\partial_{\nu}\left(g_{\mu,\tau}\right)\right)+\partial_{\alpha}\left(\partial_{\mu}\left(g_{\nu,\tau}\right)\right)-\left(\partial_{\alpha}\left(\partial_{\tau}\left(g_{\mu,\nu}\right)\right)\right)\right)}{2} \\ & - \frac{\left(\partial_{\nu}\left(g^{\alpha,\lambda}\right)\right)\left(\partial_{\mu}\left(g_{\alpha,\lambda}\right)\right)}{2} - \frac{g^{\alpha,\lambda}\left(\partial_{\nu}\left(\partial_{\mu}\left(g_{\alpha,\lambda}\right)\right)\right)}{2} \\ & + \frac{g^{\beta,\chi}\left(\partial_{\nu}\left(g_{\chi,\mu}\right)+\partial_{\mu}\left(g_{\chi,\nu}\right)-\left(\partial_{\chi}\left(g_{\mu,\nu}\right)\right)\right)g^{\alpha,\kappa}\left(\partial_{\beta}\left(g_{\alpha,\kappa}\right)\right)}{4} - \frac{1}{4}\Big(\\ & g^{\beta,\upsilon}\left(\partial_{\mu}\left(g_{\alpha,\upsilon}\right)+\partial_{\alpha}\left(g_{\mu,\upsilon}\right)-\left(\partial_{\upsilon}\left(g_{\alpha,\mu}\right)\right)\right)g^{\alpha,\sigma}\left(\partial_{\nu}\left(g_{\beta,\sigma}\right)+\partial_{\beta}\left(g_{\nu,\sigma}\right)\right. \\ & \left.-\left(\partial_{\sigma}\left(g_{\beta,\nu}\right)\right)\right)\Big)\Bigg) \end{aligned} \tag{8.1.16}$$

And in this particular case,

> *Define*(**(8.1.16)**)

Defined objects with tensor properties

$$\left\{\mathscr{D}_{\mu}, \gamma_{\mu}, K^{\mu}, \sigma_{\mu}, R_{\mu,\nu}, R_{\mu,\nu,\alpha,\beta}, T_{\mu,\nu}, C_{\mu,\nu,\alpha,\beta}, X_{\mu}, \partial_{\mu}, e_{a,\mu}, \eta_{a,b}, g_{\mu,\nu}, \gamma_{a,b,c}, l_{\mu}, \lambda_{a,b,c}, m_{\mu}, \overline{m}_{\mu}, n_{\mu}, \Gamma_{\mu,\nu,\alpha}, G_{\mu,\nu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}\right\} \tag{8.1.17}$$

Check the components:

> *T*[ ]

$$T_{\mu, \nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(8.1.18)}$$

One could query the database, directly from the spacetime metrics command (g_), about the solutions (metrics) to Einstein's equations related to Levi-Civita, the Italian mathematician

> *g_*[*civi*]

[12, 16, 1] = ["Authors" = ["Bertotti (1959)", "Kramer (1978)", "Levi-Civita (1917)", "Robinson (1959)"], "PrimaryDescription" = "EinsteinMaxwell", "SecondaryDescription" = ["Homogeneous"]]

[12, 18, 1] = ["Authors" = ["Bertotti (1959)", "Kramer (1978)", "Levi-Civita (1917)", "Robinson (1959)"], "PrimaryDescription" = "EinsteinMaxwell", "SecondaryDescription" = ["Homogeneous"]]

[12, 19, 1] = ["Authors" = ["Bertotti (1959)", "Kramer (1978)", "Levi-Civita (1917)", "Robinson (1959)"], "PrimaryDescription" = "EinsteinMaxwell", "SecondaryDescription" = ["Homogeneous"], "Comments" = ["_lambda=_zeta"]]

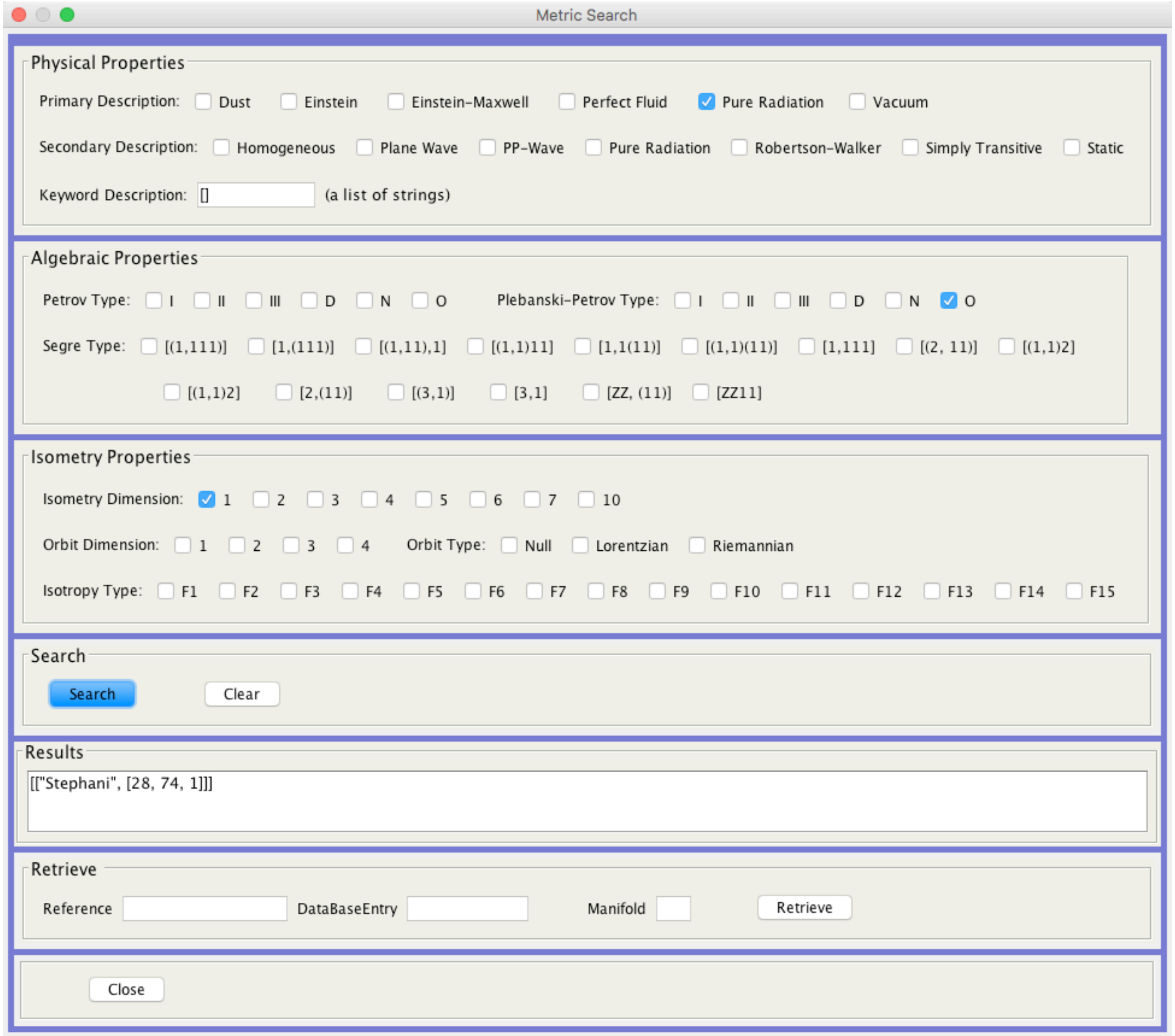
[22, 7, 1] = ["Authors" = ["Levi-Civita (1917), Frehland (1971)"], "PrimaryDescription" = "Vacuum", "SecondaryDescription" = ["Cylindrically-Symmetric"], "Comments" = ["Locally static, Weyl class_m=0,1 - flat, _m=1/2, 2, -1 - PetrovType D"]] **(8.1.19)**

These solutions can be set in one go from the metrics command, just by indicating the number with which it appears in the "Exact Solutions to Einstein's Equations" book.

The ability to query rapidly, set things in one go, change everything again etc. is at this point fantastic. The search can also be done visually, by properties.

Example: Search for solutions in the database that are of *Pure Radiation* type, with *Petrov Type* "*D*", *Plebanski-Petrov Type* "*O*" and that have *Isometry Dimension* equal to 1:

> *DifferentialGeometry:-Library:-MetricSearch*( )



Set the solution and everything related in one go

> g_[[28, 74, 1]]

Systems of spacetime Coordinates are: $\{X = (u, \eta, r, y)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (u, \eta, r, y)\}$

The Frolov and Khlebnikov (1975) metric in coordinates $[u, \eta, r, y]$

Parameters: $[\kappa 0, m(u), b, d]$

Comments: With _m(u) = constant, the metric is Ricci flat and becomes 28.24 in Stephani.

Resetting the signature of spacetime from "- - - +" to `- + + +` in order to match the signature in the database of metrics:

$$g_{\mu,\nu} = \left[\left[\frac{2\, m(u)^3 - 6\, m(u)^2\, \eta\, r - r^2\, \left(-6\,\eta^2 + b\right)\, m(u) + r^3\, \left(-2\,\eta^3 + b\,\eta + d\right)}{r\, m(u)^2},\right.\right. \quad \textbf{(8.1.20)}$$

$$- \frac{r^2}{m(u)}, -1, 0 \Big],$$

$$\Big[- \frac{r^2}{m(u)}, \frac{r^2}{-2\eta^3 + b\eta + d}, 0, 0 \Big],$$

$$\Big[-1, 0, 0, 0 \Big],$$

$$\Big[0, 0, 0, r^2 \left(-2\eta^3 + b\eta + d\right) \Big] \Big]$$

- ***100% of the solutions collected in Exact Solutions to Einstein's Field Equations were digitized***
- ***The Physics package has 76 computational commands to work with these spacetime metric solutions, plus a programming language for Physics with 175 commands***
- ***The DifferentialGeometry package has 289 commands to work with these solutions***
- ***The PDEtools package has other 56 commands to tackle diverse aspects of the equivalence problem for these solutions***

Concluding remark, how many solutions are in the book?

> *DifferentialGeometry*:-*Library*:-*Retrieve*("Stephani", 1)

[[8, 33, 1], [8, 34, 1], [12, 6, 1], [12, 7, 1], [12, 8, 1], [12, 8, 2], [12, 8, 3], [12, 8, 4], [12, 8, 5], [12, 8, 6], [12, 8, 7], [12, 8, 8], [12, 9, 1], [12, 9, 2], [12, 9, 3], [12, 12, 1], [12, 12, 2], [12, 12, 3], [12, 12, 4], [12, 13, 1], [12, 14, 1], [12, 16, 1], [12, 18, 1], [12, 19, 1], [12, 21, 1], [12, 23, 1], [12, 23, 2], [12, 23, 3], [12, 24.1, 1], [12, 24.2, 1], [12, 24.3, 1], [12, 26, 1], [12, 27, 1], [12, 28, 1], [12, 29, 1], [12, 30, 1], [12, 31, 1], [12, 32, 1], [12, 34, 1], [12, 35, 1], [12, 36, 1], [12, 37, 1], [12, 37, 2], [12, 37, 3], [12, 37, 4], [12, 37, 5], [12, 37, 6], [12, 37, 7], [12, 37, 8], [12, 37, 9], [12, 38, 1], [12, 38, 2], [12, 38, 3], [12, 38, 4], [12, 38, 5], [13, 2, 1], [13, 2, 2], [13, 2, 3], [13, 7, 1], [13, 7, 2], [13, 7, 3], [13, 7, 4], [13, 7, 5], [13, 7, 6], [13, 7, 7], [13, 7, 8], [13, 14, 1], [13, 14, 2], [13, 14, 3], [13, 19, 1], [13, 31, 1], [13, 32, 1], [13, 46, 1], [13, 48, 1], [13, 49, 1], [13, 49, 2], [13, 51, 1], [13, 53, 1], [13, 59, 1], [13, 59, 2], [13, 60, 1], [13, 60, 2], [13, 60, 3], [13, 60, 4], [13, 60, 5], [13, 60, 6], [13, 60, 7], [13, 60, 8], [13, 61, 1], [13, 61, 2], [13, 62, 1], [13, 62, 2], [13, 62, 4], [13, 62, 6], [13, 63, 1], [13, 63, 2], [13, 63, 3], [13, 63, 4], [13, 64, 1], [13, 64, 2], [13, 64, 3], [13, 64, 4], [13, 65, 1], [13, 69, 1], [13, 71, 1], [13, 72, 1], [13, 73, 1], [13, 74, 1], [13, 74, 2], [13, 74, 3], [13, 76, 1], [13, 77, 1], [13, 77, 2], [13, 79, 1], [13, 79, 2], [13, 80, 1], [13, 81, 1], [13, 83, 1], [13, 84, 1], [13, 84, 2], [13, 84, 3], [13, 85, 1], [13, 85, 2], **(8.1.21)**

[13, 86, 1], [13, 87, 1], [14, 6.1, 1], [14, 6.2, 1], [14, 6.3, 1], [14, 7, 1], [14, 8.1, 1], [14, 8.2, 1], [14, 8.3, 1], [14, 9.1, 1], [14, 9.2, 1], [14, 10, 1], [14, 10, 2], [14, 12, 1], [14, 12, 2], [14, 12, 3], [14, 14, 1], [14, 14, 2], [14, 15, 1], [14, 15.1, 2], [14, 15.2, 2], [14, 15.3, 2], [14, 16, 1], [14, 16, 2], [14, 17, 1], [14, 18, 1], [14, 18, 2], [14, 19, 1], [14, 20, 1], [14, 21, 1], [14, 21, 2], [14, 21, 3], [14, 22, 1], [14, 23, 1], [14, 24, 1], [14, 25, 1], [14, 26, 1], [14, 26, 2], [14, 26, 3], [14, 26, 4], [14, 27, 1], [14, 28, 1], [14, 28, 2], [14, 28, 3], [14, 29, 1], [14, 30, 1], [14, 31, 1], [14, 32, 1], [14, 33, 1], [14, 35, 1], [14, 37, 1], [14, 38, 1], [14, 38, 2], [14, 38, 3], [14, 39, 1], [14, 39, 2], [14, 39, 3], [14, 39, 4], [14, 39, 5], [14, 39, 6], [14, 40, 1], [14, 41, 1], [14, 42, 1], [14, 46, 1], [15, 3, 1], [15, 3, 2], [15, 4, 1], [15, 4, 2], [15, 4, 3], [15, 9, 1], [15, 10, 1], [15, 12, 1], [15, 12, 2], [15, 12, 3], [15, 12, 4], [15, 12, 5], [15, 12, 6], [15, 17, 1], [15, 17, 2], [15, 17, 3], [15, 17, 4], [15, 18, 1], [15, 19, 1], [15, 19, 2], [15, 20, 1], [15, 21, 1], [15, 21, 2], [15, 22, 1], [15, 23, 1], [15, 23, 2], [15, 24, 1], [15, 24, 2], [15, 25, 1], [15, 25, 2], [15, 26, 1], [15, 26, 2], [15, 27, 1], [15, 27, 2], [15, 27, 3], [15, 27, 4], [15, 27, 5], [15, 27, 6], [15, 27, 7], [15, 27, 8], [15, 28, 1], [15, 29, 1], [15, 30, 1], [15, 31, 1], [15, 32, 1], [15, 34, 1], [15, 34, 2], [15, 34, 3], [15, 43, 1], [15, 43, 2], [15, 43, 3], [15, 50, 1], [15, 50, 2], [15, 50, 3], [15, 50, 4], [15, 50, 5], [15, 50, 6], [15, 62, 1], [15, 62, 2], [15, 62, 3], [15, 63, 1], [15, 63, 2], [15, 63, 3], [15, 65, 1], [15, 65, 2], [15, 66, 1], [15, 66, 2], [15, 66, 3], [15, 75, 1], [15, 75, 2], [15, 75, 3], [15, 77, 1], [15, 77, 2], [15, 77, 3], [15, 78, 1], [15, 79, 1], [15, 81, 1], [15, 81, 2], [15, 81, 3], [15, 82, 1], [15, 82, 2], [15, 82, 3], [15, 83, 1.1], [15, 83, 1.2], [15, 83, 2], [15, 83, 3.1], [15, 83, 3.2], [15, 83, 4], [15, 84, 1], [15, 85, 1], [15, 85, 2], [15, 85, 3], [15, 86, 1], [15, 86, 2], [15, 86, 3], [15, 87, 1], [15, 87, 2], [15, 87, 3], [15, 87, 4], [15, 87, 5], [15, 88, 1], [15, 89, 1], [15, 90, 1], [16, 1, 1], [16, 1, 2], [16, 1, 3], [16, 1, 4], [16, 1, 5], [16, 1, 6], [16, 1, 7], [16, 1, 8], [16, 1, 9], [16, 1, 10], [16, 1, 11], [16, 1, 12], [16, 1, 13], [16, 1, 14], [16, 1, 15], [16, 1, 16], [16, 1, 17], [16, 1, 18], [16, 1, 19], [16, 1, 20], [16, 1, 21], [16, 1, 22], [16, 1, 23], [16, 1, 24], [16, 1, 25], [16, 1, 26], [16, 1, 27], [16, 14, 1], [16, 14, 2], [16, 14, 3], [16, 14, 4], [16, 14, 5], [16, 14, 6], [16, 14, 7], [16, 14, 8], [16, 14, 9], [16, 14, 10], [16, 14, 11], [16, 14, 12], [16, 14, 13], [16, 14, 14], [16, 14, 15], [16, 14, 16], [16, 14, 17], [16, 14, 18], [16, 14, 19], [16, 14, 20], [16, 18, 1], [16, 19, 1], [16, 20, 1], [16, 22, 1], [16, 24, 1], [16, 24, 2], [16, 43, 1], [16, 45, 1], [16, 45, 2], [16, 46, 1], [16, 47, 1], [16, 50, 1], [16, 51, 1], [16, 54, 1], [16, 61, 1], [16, 63, 1], [16, 66, 1], [16, 66, 2], [16, 66, 3], [16, 67, 1], [16, 71, 1], [16, 72, 1], [16, 73, 1], [16, 74, 1], [16, 75, 1], [16, 76, 1], [16, 77, 1], [16, 77, 2], [16, 77, 3], [16, 78, 1], [17, 4, 1], [17, 4, 2], [17, 5, 1], [17, 9, 1], [17, 14, 1], [17, 15, 1], [17, 15, 2], [17, 16, 1], [17, 20, 1], [17, 22, 1], [17, 23, 1], [17, 24, 1], [17, 24, 2], [17, 26, 1], [17, 27, 1], [17, 27, 2], [17, 28, 1], [17, 28, 2], [17, 29, 1], [17, 29, 2], [17, 30, 1], [17, 31, 1], [18, 2, 1], [18, 2, 2], [18, 2, 3], [18, 2, 4], [18, 2, 5], [18, 2, 6], [18, 2, 7], [18, 2, 8], [18, 48, 1], [18, 48, 2], [18, 49, 1], [18, 50, 1], [18, 64, 1], [18, 64, 2], [18, 64, 3], [18, 65, 1], [18, 66, 1], [18, 67, 1], [18, 71, 1], [18, 75, 1], [19, 17, 1], [19, 17, 2], [19, 21, 1], [20, 3, 1], [20, 4, 1], [20, 5, 1], [20, 7, 1], [20, 8, 1], [20, 9, 1], [20, 10, 1], [20, 11, 1], [20, 12, 1], [20, 13, 1], [20, 15, 1], [20, 16, 1], [20, 17, 1], [20, 20, 1], [20, 21, 1], [20, 23, 1], [20, 27, 1], [20, 28, 1], [20, 29, 1], [20, 32, 1], [20, 34, 1], [20, 36, 1], [20, 38, 1], [20, 38, 2], [20, 38, 3], [20, 44, 1], [20, 46, 1], [20, 54, 1], [20, 57, 1], [20, 57, 2], [21, 1, 1], [21, 1, 2], [21, 1, 3], [21, 4, 1], [21, 5, 1], [21, 6, 1], [21, 7, 1], [21, 10, 1], [21, 10, 2], [21, 11, 1], [21, 16, 1], [21, 17, 1], [21, 17, 2], [21, 20, 1], [21, 22, 1], [21, 22, 2], [21, 24, 1], [21, 28, 1], [21, 30, 1], [21, 30, 2], [21, 30, 3], [21, 31, 1], [21, 35, 1], [21, 41, 1], [21, 52, 1], [21, 57, 1], [21, 58, 1], [21, 59, 1], [21, 60, 1], [21, 61, 1], [21, 61, 2], [21, 61, 3], [21, 61, 4], [21, 61, 5], [21, 70, 1], [21, 71, 1], [21, 72, 1], [21, 73, 1], [21, 74, 1], [21, 74, 2], [21, 74, 3], [21, 74, 4], [22, 3, 1], [22, 4, 1], [22, 4, 2], [22, 5, 1], [22, 6, 1], [22, 6, 2], [22, 7, 1], [22, 8, 1], [22, 8, 2], [22, 8, 3], [22, 8, 4], [22, 8, 5], [22, 11, 1], [22, 12, 1], [22, 13, 1], [22, 14, 1], [22, 15, 1], [22, 16, 1], [22, 17, 1], [22, 18, 1], [22, 18, 2], [22, 19, 1], [22, 22, 1], [22, 23, 1], [22, 24, 1], [22, 27, 1], [22, 28, 1], [22, 29, 1], [22, 34, 1], [22, 34, 2], [22, 34, 3], [22, 34, 4], [22, 34, 5], [22, 47, 1], [22, 48, 1], [22, 49, 1], [22, 50, 1], [22, 51, 1], [22, 52, 1], [22, 53, 1], [22, 59, 1], [22, 63, 1], [22, 64, 1], [22, 67, 1], [22, 67, 2], [22, 70, 1], [23, 1, 1], [23, 2, 1], [23, 2, 2], [23, 2, 3], [23, 2, 4], [23, 3, 1], [23, 4, 1], [23, 6, 1], [23, 6, 2], [23, 7, 1], [23, 8, 1], [23, 8, 2], [23, 8, 3], [23, 11, 1], [23, 12, 1], [23, 13, 1], [23, 13, 2], [23, 13, 3], [23, 13, 4], [23, 14, 1], [23, 14, 2], [23, 14, 3], [23, 14, 4], [23, 14, 5], [23, 14, 6], [23, 14, 7], [23, 15, 1], [23, 15, 2], [23, 15, 3], [23, 15, 4], [23, 15, 5], [23, 16, 1], [23, 16, 2], [23, 16, 3], [23, 16, 4], [23, 16, 5], [23, 17, 1], [23, 17, 2], [23, 18, 1], [23, 18, 2], [23, 19, 1], [23, 19, 2], [23, 20, 1], [23, 20, 2], [23, 20, 3], [23, 21, 1], [23, 21, 2], [23, 22, 1], [23, 23, 1], [23, 24, 1], [23, 25, 1], [23, 26, 1], [23, 26, 2], [23, 26, 3], [23, 26, 4], [23, 26, 5], [23, 26, 6], [23, 26, 7], [23, 26, 8], [23, 30, 1], [23, 31, 1], [23, 32, 1], [23, 32, 2], [23, 32, 3], [23, 33, 1], [23, 33, 2], [23, 33, 3], [23, 34, 1], [23, 35, 1], [23, 36, 1], [23, 36, 2], [23, 36, 3], [23, 37, 1], [23, 38, 1], [23, 38, 2], [23, 38, 3], [23, 39, 1], [23, 39, 2], [23, 40, 1], [23, 40, 2], [23, 40, 3], [23, 40, 4], [23, 41, 1], [23, 41, 2], [23, 41, 3], [23, 42, 1], [23, 43, 1], [23, 44, 1], [23, 44, 2], [23, 45, 1], [23, 46, 1], [23, 47, 1], [23, 48, 1], [23, 48, 2], [23, 49, 1], [23, 50, 1], [23, 51, 1], [23, 52, 1], [23, 53, 1], [24, 2, 1], [24, 2, 2], [24, 2, 3], [24, 2, 4], [24, 2, 5], [24, 2, 6], [24, 2, 7], [24, 2, 8], [24, 2, 9], [24, 2, 10], [24, 2, 11], [24, 21, 1], [24, 22, 1], [24, 28, 1], [24, 35, 1], [24, 37, 1], [24, 37, 2], [24, 37, 3], [24, 37, 4], [24, 37, 5], [24, 37, 6], [24, 37, 7], [24, 37, 8], [24, 37, 9], [24, 38, 1], [24, 38, 2], [24, 40, 1], [24, 46, 1], [24, 46, 2], [24, 46, 3], [24, 47, 1], [24, 51, 1], [25, 2, 1], [25, 5, 1], [25, 5, 2], [25, 6, 1], [25, 9, 1], [25, 12, 1], [25, 16, 1], [25, 16, 2], [25, 16, 3], [25, 16, 4], [25, 22, 1], [25, 24, 1], [25, 26, 1], [25, 26, 2], [25, 26, 3], [25, 28, 1], [25, 30, 1], [25, 31, 1], [25, 35, 1], [25, 36, 1], [25, 36, 2], [25, 39, 1], [25, 43, 1], [25, 45, 1], [25, 55, 1], [25, 56, 1], [25, 56, 2], [25, 57, 1], [25, 57, 2], [25, 59, 1], [25, 61, 1], [25, 62, 1], [25, 65, 1], [25, 68, 1], [25, 68, 2], [25, 68, 3], [25, 74, 1], [26, 5, 1], [26, 5, 2], [26, 5, 3], [26, 5, 4], [26, 6, 1], [26, 11, 1], [26, 13, 1], [26, 14, 1], [26, 14, 2], [26, 15, 1], [26, 16, 1], [26, 16, 2], [26, 16, 3], [26, 16, 4], [26, 16, 5], [26, 16, 6], [26, 19, 1], [26, 21, 1], [26, 22, 1], [26, 23, 1], [26, 25, 1], [26, 25, 2], [26, 25, 3], [27, 27, 1], [27, 37, 1], [28, 8, 1], [28, 12, 1], [28, 16, 1], [28, 17, 1], [28, 21, 1], [28, 21, 2], [28, 21, 3], [28, 21, 4], [28, 21, 5], [28, 21, 6], [28, 21, 7], [28, 24, 1], [28, 25, 1], [28, 26, 1], [28, 26, 2], [28, 26, 3], [28, 37, 1], [28, 41, 1], [28, 42, 1], [28, 43, 1], [28, 44, 1], [28, 44, 2], [28, 44, 3], [28, 44, 4], [28, 44, 5], [28, 44, 6], [28, 45, 1], [28, 45, 2], [28, 46, 1], [28, 46, 2], [28, 47, 1], [28, 50, 1], [28, 53, 1], [28, 53, 2], [28, 55, 1], [28, 55, 2], [28, 56.1, 1], [28, 56.2, 2], [28, 56.2, 3], [28, 56.3, 1], [28, 56.4, 1], [28, 56.5, 1], [28, 56.6, 1], [28, 58.2, 1], [28, 58.3, 1], [28, 58.3, 2], [28, 58.4, 1], [28, 59, 1], [28, 60, 1], [28, 61, 1], [28, 64, 1], [28, 66, 1], [28, 67, 1], [28, 68, 1], [28, 71, 1], [28, 72, 1], [28, 73, 1], [28, 74, 1], [28, 78, 1], [29, 13, 1], [29, 34, 1], [29, 38, 1], [29, 46, 1], [29, 53, 1], [29, 60, 1], [29, 62, 1], [29, 64, 1], [29, 71, 1], [29, 74, 1], [29, 75, 1], [30, 14, 1], [30, 20, 1], [30, 22, 1], [30, 26, 1], [30, 27, 1], [30, 27, 2], [30, 28, 1], [30, 28, 2], [30, 28, 3], [30, 30, 1], [30, 33, 1], [30, 34, 1], [30, 36, 1], [30, 37, 1], [30, 43, 1], [30, 45, 1], [30, 46, 1], [30, 47, 1], [30, 50, 1], [30, 51, 1], [30, 51, 2], [30, 51, 3], [30, 52, 1], [30, 58, 1], [30, 59, 1], [30, 62, 1], [30, 64, 1], [30, 65, 1], [30, 69, 1], [30, 71, 1], [30, 73, 1], [30, 76, 1], [31, 7, 1], [31, 8, 1], [31, 26, 1], [31, 34, 1], [31, 34, 2], [31, 38, 1], [31, 40, 1], [31, 41, 1], [31, 43, 1], [31, 43, 2], [31, 43, 3], [31, 49, 1], [31, 50, 1], [31, 56, 1], [31, 57, 1], [31, 58, 1], [31, 60, 1], [31, 61, 1], [32, 31, 1], [32, 43, 1], [32, 47, 1], [32, 59, 1], [32, 60, 1], [32, 62, 1], [32, 71, 1], [32, 78, 1], [32, 80, 1], [32, 80, 2], [32, 80, 3], [32, 80, 4], [32, 94, 1], [32, 96, 1], [32, 96, 2], [32, 99, 1], [32, 102, 1], [32, 102, 2], [32, 104, 1], [33, 5, 1], [33, 7, 1], [33, 8, 1], [33, 8, 2], [33, 8, 3], [33, 9, 1], [33, 10, 1], [33, 10, 2], [33, 11, 1], [33, 12, 1], [33, 12, 2], [33, 13, 1], [33, 14, 1], [33, 15, 1], [33, 16, 1], [33, 17, 1], [33, 18, 1], [33, 19, 1], [33, 20, 1], [33, 22, 1], [33, 23, 1], [33, 25, 1], [33, 25, 2], [33, 28, 1], [33, 30, 1], [33, 31, 1], [33, 34, 1], [33, 35, 1], [33, 38, 1], [33, 40, 1], [33, 43, 1], [33, 44, 1], [33, 45, 1], [33, 48, 1], [33, 49, 1], [34, 23, 1], [34, 25, 1], [34, 128, 1], [35, 6, 1], [35, 7, 1], [35, 8, 1], [35, 9, 1], [35, 19, 1], [35, 29, 1], [35, 30, 1], [35, 33, 1], [35, 34, 1], [35, 35, 1], [35, 73, 1], [35, 74, 1], [35, 75, 1], [35, 76, 1], [35, 76, 2], [35, 77, 1], [35, 78, 1], [35, 79, 1], [35, 80, 1], [36, 11, 1], [36, 12, 1], [36, 13, 1], [36, 14, 1], [36, 15, 1], [36, 18, 1], [36, 18, 2], [36, 19, 1], [36, 20, 1], [36, 22, 1], [36, 23, 1], [36, 24, 1], [36, 25, 1], [36, 26, 1], [36, 28, 1], [36, 30, 1], [36, 31, 1], [36, 32, 1], [36, 33, 1], [36, 34, 1], [36, 35, 1], [36, 36, 1], [36, 37, 1], [37, 13, 1], [37, 21, 1], [37, 39, 1], [37, 40, 1], [37, 45, 1], [37, 49, 1], [37, 50, 1], [37, 51, 1], [37, 53, 1], [37, 55, 1], [37, 57, 1], [37, 58, 1], [37, 58, 2], [37, 58, 3], [37, 64, 1], [37, 65, 1], [37, 66, 1], [37, 68, 1], [37, 83, 1], [37, 84, 1], [37, 98, 1], [37, 104, 1], [37, 106, 1], [38, 1, 1], [38, 2, 1], [38, 3, 1], [38, 4, 1], [38, 5, 1], [38, 6, 1]]

> *nops*(**(8.1.21)**)

971 **(8.1.22)**

References

- Landau, L.D., and Lifshitz, E.M. **The Classical Theory of Fields**, **Course of Theoretical Physics Volume 2**, fourth revised English edition. Elsevier, 1975.

- Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C. Herlt, E. **Exact Solutions of Einstein's Field Equations**, Cambridge Monographs on Mathematical Physics, second edition. Cambridge University Press, 2003.

- Cheb-Terrab, E.S. "Exact solutions to Einstein's equations", Mapleprimes blog, October 2015.

- Cheb-Terrab, E.S. "The Physics project at Maplesoft", Mapleprimes blog, July 2013.