Differential algebra with mathematical functions, symbolic powers and anticommutative variables

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Abstract:
Computer algebra implementations of Differential Algebra typically require that the systems of equations to be tackled be rational in the independent and dependent variables and their partial derivatives, and of course that $A\,B = B\,A$, everything is commutative.

It is possible, however, to extend this computational domain and apply Differential Algebra techniques to systems of equations that involve arbitrary compositions of mathematical functions (elementary or special), fractional and symbolic powers, as well as anticommutative variables and functions. This is the subject of this presentation, with examples of the implementation of these ideas in the Maple computer algebra system and its ODE and PDE solvers.

> $restart;\, with(PDEtools) : interface(imaginaryunit = i) :$

> $$sys := \left[\frac{\partial^2}{\partial y^2}\,\xi(x,y) = 0,\ -6\left(\frac{\partial}{\partial y}\,\xi(x,y)\right) y + \frac{\partial^2}{\partial y^2}\,\eta(x,y) - 2\left(\frac{\partial^2}{\partial x \partial y}\,\xi(x,y)\right) = 0,\right.$$
> $$-12\left(\frac{\partial}{\partial y}\,\xi(x,y)\right) a^2\, y - 9\left(\frac{\partial}{\partial y}\,\xi(x,y)\right) a\, y^2 - 3\left(\frac{\partial}{\partial y}\,\xi(x,y)\right) b - 3\left(\frac{\partial}{\partial x}\,\xi(x,y)\right) y$$
> $$- 3\,\eta(x,y) + 2\left(\frac{\partial^2}{\partial x \partial y}\,\eta(x,y)\right) - \left(\frac{\partial^2}{\partial x^2}\,\xi(x,y)\right) = 0,\ -8\left(\frac{\partial}{\partial x}\,\xi(x,y)\right) a^2\, y$$
> $$- 6\left(\frac{\partial}{\partial x}\,\xi(x,y)\right) a\, y^2 + 4\left(\frac{\partial}{\partial y}\,\eta(x,y)\right) a^2\, y + 3\left(\frac{\partial}{\partial y}\,\eta(x,y)\right) a\, y^2 - 4\,\eta(x,y)\, a^2$$
> $$- 6\,\eta(x,y)\, a\, y - 2\left(\frac{\partial}{\partial x}\,\xi(x,y)\right) b + \left(\frac{\partial}{\partial y}\,\eta(x,y)\right) b - 3\left(\frac{\partial}{\partial x}\,\eta(x,y)\right) y + \frac{\partial^2}{\partial x^2}$$
> $$\left.\eta(x,y) = 0\right] :$$

> $declare((\xi, \eta)(x, y))$

$\xi(x,y)$ *will now be displayed as* ξ

$\eta(x,y)$ *will now be displayed as* η **(1)**

> **for** *eq* **in** *sys* **do** *eq* **od**;

$$\xi_{y,y} = 0$$

$$-6\,\xi_y\, y + \eta_{y,y} - 2\,\xi_{x,y} = 0$$

$$-12\,\xi_y\, a^2\, y - 9\,\xi_y\, a\, y^2 - 3\,\xi_y\, b - 3\,\xi_x\, y - 3\,\eta + 2\,\eta_{x,y} - \xi_{x,x} = 0$$

(2)

$$-8\,\xi_x\,a^2\,y-6\,\xi_x\,a\,y^2+4\,\eta_y\,a^2\,y+3\,\eta_y\,a\,y^2-4\,\eta\,a^2-6\,\eta\,a\,y-2\,\xi_x\,b+\eta_y\,b-3\,\eta_x\,y+\eta_{x,x}=0 \quad \textbf{(2)}$$

> *casesplit*(*sys*)

$$[\eta=0,\xi_x=0,\xi_y=0]\ \&\text{where}\ [\ ] \quad \textbf{(3)}$$

>

Differential polynomial forms for mathematical functions (basic)

Consider the exponential and the square-root functions, both non-polynomial objects which however admit a differential polynomial representation:

> $g(x)=\exp(x)$

$$g(x)=e^x \quad \textbf{(4)}$$

> *dpolyform*(**(4)**)

$$[g(x)-\_F1(x)=0,\_F1_x-\_F1(x)=0],[\_F1(x)\neq 0],[\_F1(x)=e^x] \quad \textbf{(5)}$$

In the above there is a sequence of 3 lists. The first list contains the "Equations" of the problem; the second list contains the "Inequations" and the third list contains the "back-substitution equations", telling which non-polynomial object is represented by each auxiliary function _Fn.

> $f(x)=\text{sqrt}(x)$

$$f(x)=\sqrt{x} \quad \textbf{(6)}$$

> *dpolyform*(**(6)**)

$$[f(x)-\_F1(x)=0,\_F1(x)^2-x=0],[\_F1(x)\neq 0],[\_F1(x)=\sqrt{x}] \quad \textbf{(7)}$$

>

Differential polynomial forms for compositions of mathematical functions

What happens with their composition?

> $ee:=h(x)=\exp(\text{sqrt}(x))$

$$ee:=h(x)=e^{\sqrt{x}} \quad \textbf{(8)}$$

> *dpolyform*(**(8)**)

$$[h(x)-\_F1(x)=0,2\,\_F2(x)\,\_F1_x-\_F1(x)=0,\_F2(x)^2-x=0],[\_F1(x)\neq 0,\_F2(x)\neq 0],[\_F2(x)=\sqrt{x},\_F1(x)=e^{\sqrt{x}}] \quad \textbf{(9)}$$

_F2(x) is an auxiliary function introduced to represent the (inner) sqrt.

The non-obvious equation of the first list is obtained by changing variables: from the differential polynomial form for the (outter) exponential

> *dpolyform*(**(4)**, *no_Fn*)

(10)

$$[g_x = g(x)] \,\&\text{where}\, [g(x) \neq 0] \quad \textbf{(10)}$$

> $tr := \{x = \mathrm{sqrt}(u), g(x) = h(u)\}$

$$tr := \{x = \sqrt{u}, g(x) = h(u)\} \quad \textbf{(11)}$$

> $dchange(tr, \textbf{(10)})$

$$[2\sqrt{u}\, h_u = h(u)] \,\&\text{where}\, [h(u) \neq 0] \quad \textbf{(12)}$$

> $subs(\mathrm{sqrt}(u) = \_F2(x), h = \_F1, u = x, \textbf{(12)})$

$$[2\,\_F2(x)\,\_F1_x = \_F1(x)] \,\&\text{where}\, [\_F1(x) \neq 0] \quad \textbf{(13)}$$

Moreover, take the list of equations in **(9)**

> $\textbf{(9)}[1]$

$$[h(x) - \_F1(x) = 0, 2\,\_F2(x)\,\_F1_x - \_F1(x) = 0, \_F2(x)^2 - x = 0] \quad \textbf{(14)}$$

The auxiliary functions ***can always be eliminated***: rank them higher than the rest:

> $casesplit(\textbf{(14)}, [\_F2, \_F1, h])$

$$\left[\_F2(x) = \frac{h(x)}{2\,h_x}, \_F1(x) = h(x), {h_x}^2 = \frac{h(x)^2}{4\,x}\right] \&\text{where}\, [h_x \neq 0], [\_F2(x)^2 = x, \_F1(x) = 0, h(x) = 0] \,\&\text{where}\, [\_F2(x) \neq 0] \quad \textbf{(15)}$$

From where the differential equation satisfied by $h(x) = e^{\sqrt{x}}$ is

> $op([1, 1, 3], [\textbf{(15)}])$

$${h_x}^2 = \frac{h(x)^2}{4\,x} \quad \textbf{(16)}$$

> $eval(\textbf{(16)}, ee)$

$$\frac{\left(e^{\sqrt{x}}\right)^2}{4\,x} = \frac{\left(e^{\sqrt{x}}\right)^2}{4\,x} \quad \textbf{(17)}$$

All this process was encoded in the Maple system in 1998.

> ee

$$h(x) = e^{\sqrt{x}} \quad \textbf{(18)}$$

> $dpolyform(ee, no\_Fn)$

$$\left[{h_x}^2 = \frac{h(x)^2}{4\,x}\right] \&\text{where}\, [h_x \neq 0] \quad \textbf{(19)}$$

Summarizing: ***if we compose algebraic blocks that admit differential polynomial representations, their composition also admits a differential polynomial representation***

Generalization to many variables

The generalization to many variables is straightforward

> $\tan\left(2\,x + y^{\frac{1}{2}}\right)$

$$\tan\left(2x+\sqrt{y}\right) \quad (20)$$

Call this a function of *x, y*

> $G := g(x,y) = \%$

$$G := g(x,y) = \tan\left(2x+\sqrt{y}\right) \quad (21)$$

To have functionality ommited from the display and derivatives displayed with indexed notation,

> $declare(g(x,y), \_F1(x,y), \_F2(x,y), \_F3(x,y))$

$g(x,y)$ *will now be displayed as* g

$\_F1(x,y)$ *will now be displayed as* $\_F1$

$\_F2(x,y)$ *will now be displayed as* $\_F2$

$\_F3(x,y)$ *will now be displayed as* $\_F3$ (22)

> $dpolyform(G)$

$$\left[g-\_F1=0, 2\,\_F2\,\_F1_y-1-\_F1^2=0, -2\,\_F1^2+\_F1_x-2=0, \_F2^2-y=0, \_F2_x=0\right], \left[\_F2\neq 0, \_F1_x\neq 0, \_F2\,\_F1_y\neq 0\right], \left[\_F2=\sqrt{y}, \_F1=\tan\left(2x+\sqrt{y}\right)\right] \quad (23)$$

> $PDE\_sys\_for\_G := dpolyform(G, no\_Fn)$

$$PDE\_sys\_for\_G := \left[g_x=2g^2+2, g_y^2=\frac{g^4}{4y}+\frac{g^2}{2y}+\frac{1}{4y}\right] \&\text{where} \left[g^2+1\neq 0, g_y\neq 0\right] \quad (24)$$

Verify that G satisfies this non-linear - however differential polynomial - PDE system

> $pdetest(G, PDE\_sys\_for\_G)$

$$[0,0] \quad (25)$$

CAVEAT: while G satisfies the differential polynomial form, the solution of the latter is more general than G:

> $pdsolve(PDE\_sys\_for\_G)$

$$\left\{g=\tan\left(2x+\sqrt{y}+2\,\_C1\right)\right\} \quad (26)$$

Moreover, due to the nonlinear character of this example, if one excludes the inequations

> $op(1, PDE\_sys\_for\_G)$

$$\left[g_x=2g^2+2, g_y^2=\frac{g^4}{4y}+\frac{g^2}{2y}+\frac{1}{4y}\right] \quad (27)$$

then **pde_sys** also admits singular solutions not related to G

> $pdsolve(\%)$

$$\{g=-\mathrm{i}\}, \{g=\mathrm{i}\}, \left\{g=\tan\left(2x+\sqrt{y}+2\,\_C1\right)\right\} \quad (28)$$

>

Arbitrary functions of algebraic expressions

> $f(x^2+g(y))$

$$f\left(x^2+g(y)\right) \quad (29)$$

Call this $F(x, y)$

> $F(x, y) = \%$

$$F(x, y) = f\left(x^2 + g(y)\right) \tag{30}$$

> $dpolyform(\%, no\_Fn)$

$$\left[F_y = \frac{F_x\, g_y}{2\, x}\right] \&\text{where } [\,] \tag{31}$$

So, we know nothing about the mapping f but, because of the algebraic structure of its dependency, we know the PDE system satisfied by $F(x, y) = f(x + g(y))$

Once the mechanism is understood, it becomes clear that one can do differential eliminination mostly every possible object, it is all about representing it first in differential polynomial form using auxiliary functions; derivatives and integrals are naturally represented the same way

> $F(x, y) = \mathrm{D}(f)\left(x^2 + g(y)\right)$

$$F(x, y) = \mathrm{D}(f)\left(x^2 + g(y)\right) \tag{32}$$

> $dpolyform(\%, no\_Fn)$

$$\left[F_y = \frac{F_x\, g_y}{2\, x}\right] \&\text{where } [\,] \tag{33}$$

> $F(x, y) = Int\left(f\left(x^2 + g(y)\right), x\right)$

$$F(x, y) = \int f\left(x^2 + g(y)\right) \mathrm{d}x \tag{34}$$

> $dpolyform(\%, no\_Fn)$

$$\left[F_{x, y} = \frac{F_{x, x}\, g_y}{2\, x}\right] \&\text{where } [F(x, y) \neq 0] \tag{35}$$

More complicated examples present no problem: *provided that, from the inner expressions to the outer ones, each mathematical block admits a differential polynomial form, the arbitrary function satisfies a differential polynomial PDE system*

> $F(x, y) = f(\exp(\mathrm{sqrt}(x)) + g(y))$

$$F(x, y) = f\left(\mathrm{e}^{\sqrt{x}} + g(y)\right) \tag{36}$$

> $dpolyform(\%, no\_Fn)$

$$\left[F_{x, y} = \frac{F_x\, F_{y, y}}{F_y} - \frac{F_x\, g_{y, y}}{g_y}, F_{x, x}^{\ 2} = \left(\frac{2\, F_x^{\ 2}\, F_{y, y}}{F_y^{\ 2}} - \frac{2\, F_x^{\ 2}\, g_{y, y}}{g_y\, F_y} - \frac{F_x}{x}\right) F_{x, x} - \frac{F_x^{\ 4}\, F_{y, y}^{\ 2}}{F_y^{\ 4}} + \left(\frac{2\, F_x^{\ 4}\, g_{y, y}}{g_y\, F_y^{\ 3}} + \frac{F_x^{\ 3}}{F_y^{\ 2}\, x}\right) F_{y, y} - \frac{F_x^{\ 4}\, g_{y, y}^{\ 2}}{g_y^{\ 2}\, F_y^{\ 2}} - \frac{F_x^{\ 3}\, g_{y, y}}{x\, g_y\, F_y} + \frac{(x-1)\, F_x^{\ 2}}{4\, x^2}\right] \&\text{where } \left[-2\, F_{y, y}\, F_x^{\ 2}\, x\, g_y + 2\, g_{y, y}\, F_x^{\ 2}\, x\, F_y + 2\, F_{x, x}\, F_y^{\ 2}\, g_y\, x + F_x\, g_y\, F_y^{\ 2} \neq 0\right] \tag{37}$$

> $pdetest(\mathbf{(36)}, \mathbf{(37)})$

(38)

$$[0, 0] \quad (38)$$

>

Examples of the use of this extension to include mathematical functions

Identities for special functions, or relations between them and simpler Liouvillian functions

> $declare(y(x), prime = x)$

$y(x)$ *will now be displayed as* y

derivatives with respect to x *of functions of one variable will now be displayed with '* **(39)**

> $a3 := y(x) = \mathrm{hypergeom}([1], [2], -2\,i\,x);$

$$a3 := y = {}_1F_1(1;\, 2;\, -2\,\mathrm{i}\,x) \quad (40)$$

A polynomial (in this case linear) ODE satisfied by **y(x)** is given by:

> $e3 := dpolyform(a3, no\_Fn)$

$$e3 := \left[y'' = \frac{(-2\,\mathrm{i}\,x - 2)\,y'}{x} - \frac{2\,\mathrm{i}\,y}{x}\right] \&\text{where } [y \neq 0] \quad (41)$$

The ODE above also admits a solution in terms of Liouvillian functions, which can be obtained by using Kovacic's algorithm. See DEtools[kovacicsols].

> $a3\_bis := dsolve(e3)[1]$

$$a3\_bis := y = \frac{\_C2\,\mathrm{e}^{-2\,\mathrm{i}\,x} + \_C1}{x} \quad (42)$$

This means that the hypergeometric function appearing in **a3** is equal to the right-hand side of **a3_bis** for some particular values of **_C1** and **_C2**.

To determine **_C1** and **_C2** , equate these expressions, expand in series and with the first terms construct a system of equations for them

> $e4 := a3 - a3\_bis :$
> $series(rhs(e4), x, 2) :$
> $sys := map(eq \rightarrow eq = 0, \{coeffs(convert(\%, polynom), x)\})$

$$sys := \{1 + 2\,\mathrm{i}\,\_C2 = 0,\ -\_C2 - \_C1 = 0\} \quad (43)$$

resulting in:

> $ans\_C := solve(sys, \{\_C1, \_C2\})$

$$ans\_C := \left\{\_C1 = -\frac{\mathrm{i}}{2},\ \_C2 = \frac{\mathrm{i}}{2}\right\} \quad (44)$$

At these values of **_C1** and **_C2**,

> $eval(e4, ans\_C)$

$$0 = {}_1F_1(1;\, 2;\, -2\,\mathrm{i}\,x) - \frac{\frac{\mathrm{i}\,\mathrm{e}^{-2\,\mathrm{i}\,x}}{2} - \frac{\mathrm{i}}{2}}{x} \quad (45)$$

from where the hypergeometric function can be isolated, resulting in the desired identity.

> $isolate(\%, \mathrm{hypergeom}([1], [2], -2\,i\,x));$

$$ {}_1F_1(1;\,2;\,-2\,\mathrm{i}\,x) = \frac{\frac{\mathrm{i}\,\mathrm{e}^{-2\,\mathrm{i}\,x}}{2} - \frac{\mathrm{i}}{2}}{x} \quad \textbf{(46)}$$

> $simplify((lhs - rhs)(\%))$

$$0 \quad \textbf{(47)}$$

>

Solving non-polynomial, non-differential systems using differential algebra

> $declare(prime = t)$

derivatives with respect to t of functions of one variable will now be displayed with ' **(48)**

> $sys := [t - \tan(y(t) + z(t) - \ln(y(t))) = 0, y(t) - \mathrm{e}^{-y(t) + z(t) + \arctan(t)} = 0]$

$$sys := [t + \tan(-y(t) - z(t) + \ln(y(t))) = 0, y(t) - \mathrm{e}^{-y(t) + z(t) + \arctan(t)} = 0] \quad \textbf{(49)}$$

We solve **sys** as follows. First compute a DPF for it

> $DP\_sys := dpolyform(sys, no\_Fn);$

$$DP\_sys := \left[y' = \frac{1}{t^2 + 1}, z' = \frac{1}{y(t)\,(t^2 + 1)}\right] \&\text{where } [y(t) + 1 \neq 0, y(t) \neq 0] \quad \textbf{(50)}$$

Second, solve **DP_sys**

> $sol\_DP\_sys := dsolve(DP\_sys, explicit)$

$$sol\_DP\_sys := \{y(t) = \arctan(t) + \_C2, z(t) = \ln(\arctan(t) + \_C2) + \_C1\} \quad \textbf{(51)}$$

This solution includes the solution of the original **sys** for some particular values of the integration constants **{_C1, _C2}**.

To determine their value, briefly, a system is built for the integration constants _C1 and _C2 by inserting this solution into the system, equating to zero, computing series, and taking the first terms

> $sys\_C := eval(sys, sol\_DP\_sys)$

$$sys\_C := [t - \tan(\arctan(t) + \_C2 + \_C1) = 0, \arctan(t) + \_C2 - \mathrm{e}^{-\_C2 + \ln(\arctan(t) + \_C2) + \_C1} = 0] \quad \textbf{(52)}$$

> $z1 := map(lhs, sys\_C):$
> $z2 := map(series, z1, t, 1):$
> $z3 := \{op(map(eq \rightarrow eq = 0, simplify(map(convert, z2, polynom))))\}$

$$z3 := \{-\tan(\_C2 + \_C1) = 0, \_C2 - \_C2\,\mathrm{e}^{-\_C2 + \_C1} = 0\} \quad \textbf{(53)}$$

Now solve for **{_C1, _C2}**:

> $sol\_C := solve(z3, \{\_C1, \_C2\}, allsolutions)$

$$sol\_C := \{\_C1 = \pi\,\_Z1, \_C2 = 0\}, \left\{\_C1 = \frac{\pi\,\_Z2}{2} + \mathrm{i}\,\pi\,\_Z3, \_C2 = \frac{\pi\,\_Z2}{2} - \mathrm{i}\,\pi\,\_Z3\right\} \quad \textbf{(54)}$$

where in Maple, by convention, **_Z1~** is an integer. The first solution is included in the second one.

The above leads to the solution for the original non-differential **sys** by directly evaluating **sol_DP_sys** at these values of the integration constants

> $sol\_sys := eval\left(sol\_DP\_sys, sol\_C_2\right)$

$$sol\_sys := \left\{y(t) = \arctan(t) + \frac{\pi\_Z2}{2} - \mathrm{i}\,\pi\_Z3, z(t) = \ln\left(\arctan(t) + \frac{\pi\_Z2}{2} - \mathrm{i}\,\pi\_Z3\right) + \frac{\pi\_Z2}{2} + \mathrm{i}\,\pi\_Z3\right\} \quad (55)$$

This solution can be verified by substituting into **sys**.

> sys

$$\left[t + \tan(-y(t) - z(t) + \ln(y(t))) = 0, y(t) - \mathrm{e}^{-y(t) + z(t) + \arctan(t)} = 0\right] \quad (56)$$

> $expand(eval(sys, sol\_sys))$

$$[0 = 0, 0 = 0] \quad (57)$$

>

Taking symbolic powers as *variables* using differential polynomial forms

Consider the following nonlinear ODE example 11 from Kamke's book, involving symbolic powers

> $declare(y(x), prime = x)$

$$y(x) \text{ will now be displayed as } y$$

$$\text{derivatives with respect to } x \text{ of functions of one variable will now be displayed with } ' \quad (58)$$

> $ode_{11} := \frac{d^2}{dx^2} y(x) + a\, x^r\, y(x)^n = 0$

$$ode_{11} := y'' + a\, x^r\, y^n = 0 \quad (59)$$

> $with(DEtools, gensys)$

$$[gensys] \quad (60)$$

The PDE system satisfied by the symmetries, that is, infinitesimals $[\xi, \eta]$ of the symmetry generator, of the ODE above is given by

> $declare((\xi, \eta)(x, y))$

$$\xi(x, y) \text{ will now be displayed as } \xi$$

$$\eta(x, y) \text{ will now be displayed as } \eta \quad (61)$$

> $sys := \left[gensys\left(ode_{11}, [\xi, \eta](x, y)\right)\right]:$

> **for** _eq **in** sys **do**
    _eq = 0
end do

$$\xi_{y, y} = 0$$

$$\eta_{y,y} - 2\,\xi_{x,y} = 0$$

$$3\,\xi_y\,x^r\,y^n\,a + 2\,\eta_{x,y} - \xi_{x,x} = 0$$

$$2\,\xi_x\,x^r\,y^n\,a - \eta_y\,x^r\,y^n\,a + \frac{\eta\,a\,x^r\,y^n\,n}{y} + \frac{\xi\,a\,x^r\,r\,y^n}{x} + \eta_{x,x} = 0 \tag{62}$$

This is a second order linear PDE system, with two unknowns $\{\eta(x,y), \xi(x,y)\}$ and four equations, involving non-polynomial objects x^r and $y(x)^n$.

Its *general solution* is computed using differential polynomial representations for the symbolic powers results in

> *PDEtools:-casesplit(sys)*

$$\left[\eta = \frac{-r\,\xi\,y - 2\,\xi\,y}{n\,x - x}, \xi_x = \frac{\xi}{x}, \xi_y = 0\right] \text{\&where } [\,] \tag{63}$$

> *sol* := *pdsolve(sys)*

$$sol := \left\{\eta = -\frac{\_C1\,y\,(r+2)}{n-1}, \xi = \_C1\,x\right\} \tag{64}$$

Now, the difficult problem: are there other solutions, not included in the general solution, related to *particular values of the parameters n and r*?

Also, in this particular problem, from the form of the ODE $y'' + a\,x^r\,y^n = 0$, the case $n = 1$ is of no interest since the ODE would become linear, and so, computing its symmetries is the same as computing its solution. Add this inequation to the PDE system.

> *sys1* := [*op(sys)*, $n \neq 1$]

$$sys1 := \left[\xi_{y,y}, \eta_{y,y} - 2\,\xi_{x,y}, 3\,\xi_y\,x^r\,y^n\,a + 2\,\eta_{x,y} - \xi_{x,x}, 2\,\xi_x\,x^r\,y^n\,a - \eta_y\,x^r\,y^n\,a + \frac{\eta\,a\,x^r\,y^n\,n}{y} + \frac{\xi\,a\,x^r\,r\,y^n}{x} + \eta_{x,x}, n \neq 1\right] \tag{65}$$

Next we run a differential elimination process splitting into cases, this is what pdsolve does internally

> *casesplit(sys1, parameters* = {*n*, *r*}, *caseplot)*

========= *Pivots Legend* =========

$$p1 = n - 2$$

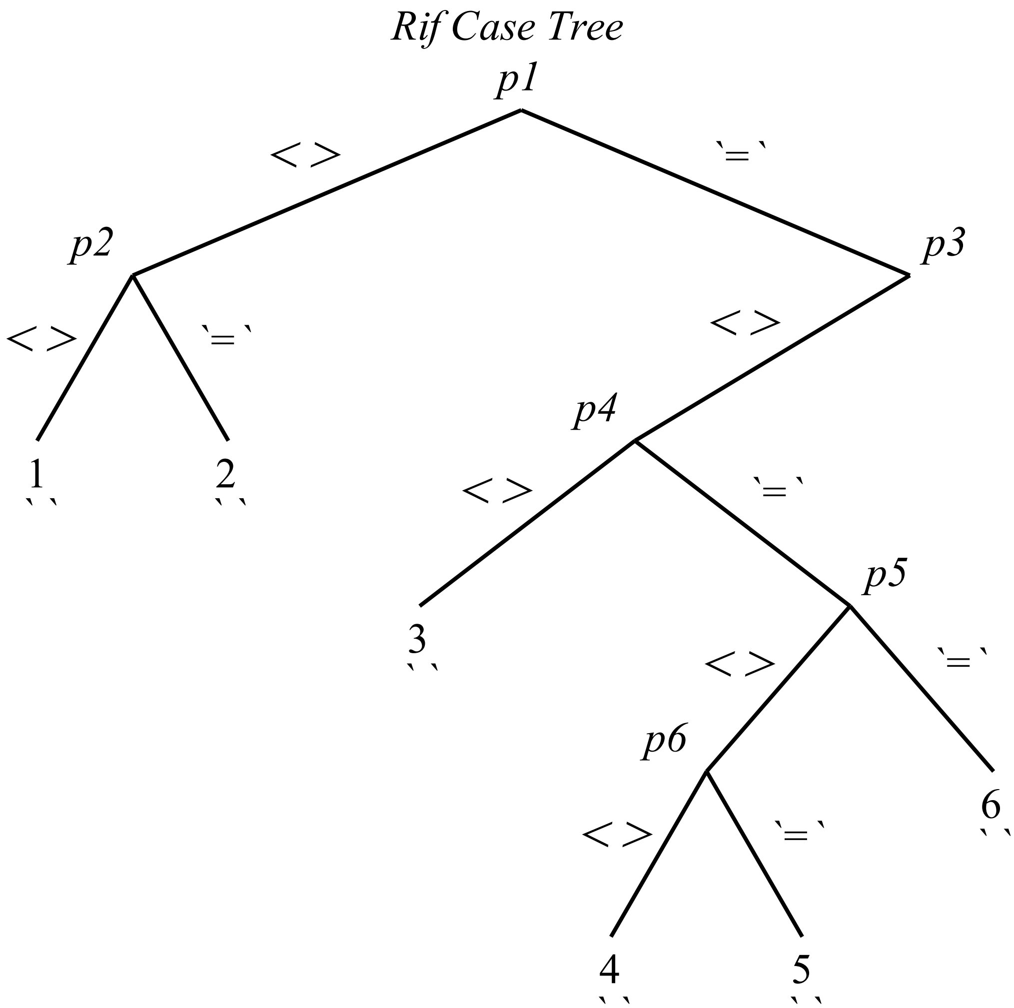
$$p2 = n + r + 3$$

$$p3 = 50\,x^r\,a\,y^n\,x^2 + y\,r\,(r+5)$$

$$p4 = 49\,r^3 + 490\,r^2 + 1525\,r + 1500$$

$$p5 = r + 5$$

$$p6 = 7\,r + 20$$



$$\left[\eta = \frac{-r\,\xi\,y - 2\,\xi\,y}{n\,x - x},\ \xi_x = \frac{\xi}{x},\ \xi_y = 0\right] \&\text{where } [n - 2 \neq 0, n - 1 \neq 0, n + r + 3 \neq 0, n \neq 0, r \neq 0], \left[\eta_x = -\frac{-\eta\,r\,x + r\,\xi\,y - 4\,\eta\,x + 2\,\xi\,y}{2\,x^2},\ \eta_y = \frac{\eta}{y},\ \xi_x = -\frac{-\eta\,r\,x + r\,\xi\,y - 4\,\eta\,x}{2\,y\,x},\ \xi_y = 0,\ n = -r - 3\right] \&\text{where } [-r - 4 \neq 0, -r - 5 \neq 0, -r - 3 \neq 0, r \neq 0], \left[\eta = \frac{-r\,\xi\,y - 2\,\xi\,y}{x},\ \xi_x = \frac{\xi}{x},\ \xi_y = 0,\ n = 2\right] \&\text{where } [r + 5 \neq 0, 7\,r + 15 \neq 0, 7\,r + 20 \neq 0, r \neq 0], \left[\eta_x = \frac{-21\,\eta\,x^r\,a\,y^n\,x + 3\,x^r\,a\,y^n\,\xi\,y}{98\,x^r\,a\,y^n\,x^2 - 12\,y},\ \eta_y = \frac{-343\,\eta\,x^r\,a\,y^n\,x^3 + 6\,\xi\,y^2}{-343\,x^r\,a\,y^n\,x^3\,y + 42\,y^2\,x},\ \xi_x = \frac{-49\,\eta\,x^r\,a\,y^n\,x^3 + 105\,x^r\,a\,y^n\,\xi\,x^2\,y - 12\,\xi\,y^2}{98\,x^r\,a\,y^n\,x^3\,y - 12\,y^2\,x},\ \xi_y = 0,\ n = 2,\ r = -\frac{15}{7}\right] \&\text{where} \tag{66}$$

$[49\,x^{r+2}\,a\,y^{n}-6\,y\neq 0]$, $\left[\eta_{x} = \frac{-98\,\eta\,x^{r}\,a\,y^{n}\,x^{3}+84\,x^{r}\,a\,y^{n}\,\xi\,x^{2}\,y-42\,\eta\,x\,y+36\,\xi\,y^{2}}{343\,x^{r}\,a\,y^{n}\,x^{4}-42\,y\,x^{2}}, \eta_{y} = \frac{-343\,\eta\,x^{r}\,a\,y^{n}\,x^{3}+36\,\xi\,y^{2}}{-343\,x^{r}\,a\,y^{n}\,x^{3}\,y+42\,y^{2}\,x}, \xi_{x} = \frac{-49\,\eta\,x^{r}\,a\,y^{n}\,x^{3}+140\,x^{r}\,a\,y^{n}\,\xi\,x^{2}\,y-12\,\xi\,y^{2}}{98\,x^{r}\,a\,y^{n}\,x^{3}\,y-12\,y^{2}\,x}, \xi_{y}=0, n=2, r=-\frac{20}{7}\right]$ &where

$[49\,x^{r+2}\,a\,y^{n}-6\,y\neq 0]$, $\left[\eta_{x}=\frac{-\eta\,x+3\,\xi\,y}{2\,x^{2}}, \eta_{y}=\frac{\eta}{y}, \xi_{x}=\frac{-\eta\,x+5\,\xi\,y}{2\,y\,x}, \xi_{y}=0, n=2, r=-5\right]$ &where [ ]

This is what pdsolve does internally, then tackling each of the PDE systems above to obtain the general and singular solutions

> *sol1* := *pdsolve*(*sys1*, *parameters* = {*n*, *r*})

$$sol1 := \{n=2, r=-5, \eta=y\,(\_C2\,x+3\,\_C1), \xi=x\,(\_C2\,x+\_C1)\}, \left\{n=2, r=-\frac{20}{7}, \eta=-\frac{2\,(-6\,x^{2}\,\_C1-98\,x^{8/7}\,\_C1\,a\,y-147\,\_C2\,a\,x\,y)}{343\,x\,a}, \xi=\_C1\,x^{8/7}+\_C2\,x\right\}, \left\{n=2, r=-\frac{15}{7}, \eta=-\frac{-49\,\_C1\,a\,x\,y-147\,x^{6/7}\,\_C2\,a\,y+12\,\_C2\,x}{343\,x\,a}, \xi=\_C1\,x+\_C2\,x^{6/7}\right\}, \{n=2, r=r, \eta=-\_C1\,y\,(r+2), \xi=\_C1\,x\}, \left\{n=-r-3, r=r, \eta=\frac{(r\,(\_C2\,x+\_C1)+4\,\_C2\,x+2\,\_C1)\,y}{r+4}, \xi=x\,(\_C2\,x+\_C1)\right\}, \left\{n=n, r=r, \eta=-\frac{\_C1\,y\,(r+2)}{n-1}, \xi=\_C1\,x\right\} \quad \textbf{(67)}$$

> *map*(*pdetest*, [*sol1*], *sys1*)

$$[[0,0,0,0],[0,0,0,0],[0,0,0,0],[0,0,0,0],[0,0,0,0],[0,0,0,0]] \quad \textbf{(68)}$$

So there exist particular values of *n* and *r* for which the system has additional solutions. The solution set with *n* and *r* integers and with xi linear in *x* is in fact a particular case of the *general solution* computed previously, but the other solution sets are not.

>

Resolving the equivalence between two solutions to Einstein's equations in the presence of special functions

> restart;

This problem is from general relativity, and amounts to found two functions $R(u, \vartheta, \varphi, v)$ and $T(u, \vartheta, \varphi, v)$ that map two apparently different forms of a Schwarzschild solution to Einstein's equations - in spherical and in Kruskal coordinates, where the relation between these coordinates is

> $tr := \left\{u = \sqrt{r - 2\,m}\ e^{\frac{r - t}{4\,m}}, v = -e^{\frac{r + t}{4\,m}} \sqrt{r - 2\,m}\right\}$

$$tr := \left\{u = \sqrt{r - 2\,m}\ e^{\frac{r - t}{4\,m}}, v = -e^{\frac{r + t}{4\,m}} \sqrt{r - 2\,m}\right\} \quad \textbf{(69)}$$

The inverse transformation involves the logarithm and LambertW functions

> $simplify(solve(tr, \{r, t\}))$

$$\left\{r = 2\left(W\left(-\frac{u\,v\,e^{-1}}{2\,m}\right) + 1\right) m, t = 2\ln\left(-\frac{v}{u}\right) m\right\} \quad \textbf{(70)}$$

> $lprint(\%)$

```
{r = 2*(LambertW(-(1/2)*u*v*exp(-1)/m)+1)*m, t = 2*ln(-v/u)*m}
```

To set the problem, define two sets of coordinates (we are mapping from spherical to Kruskal coordinates)

> $with(Physics) : with(PDEtools) :$

> $Coordinates(X = spherical)$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (r, \theta, \phi, t)\}$

Systems of spacetime Coordinates are: $\{X = (r, \theta, \phi, t)\}$

$$\{X\} \quad \textbf{(71)}$$

> $Coordinates(K = [u, \vartheta, \varphi, v])$

Systems of spacetime Coordinates are: $\{K = (u, \vartheta, \varphi, v), X = (r, \theta, \phi, t)\}$

$$\{K, X\} \quad \textbf{(72)}$$

And we will search for a transformation of the form

> $\{r = R(K), t = T(K)\}$

$$\{r = R(K), t = T(K)\} \quad \textbf{(73)}$$

Skipping the details of how we arrive at the system of equations to be solved, we have that R and T must satisfy

> $sys := \Bigg\{\left(-4\left(-\frac{1}{2}R(K) + m\right)^2 \left(\frac{\partial}{\partial u}T(K)\right)^2 + \left(\frac{\partial}{\partial u}R(K)\right)^2 R(K)^2\right)\frac{1}{-R(K) + 2\,m}\,\frac{1}{R(K)} = 0, \left(-4\left(-\frac{1}{2}R(K) + m\right)^2 \left(\frac{\partial}{\partial v}T(K)\right)^2 + \left(\frac{\partial}{\partial v}R(K)\right)^2 R(K)^2\right)\frac{1}{-R(K) + 2\,m}\,\frac{1}{R(K)} = 0, \Big(-4\left(-\frac{1}{2}R(K) + m\right)^2 \left(\frac{\partial}{\partial \varphi}T(K)\right)^2 + 2\,R(K)^2 \left(\frac{1}{2}\left(\frac{\partial}{\partial \varphi}R(K)\right)^2\right.$

$$+ \left(\cos(\vartheta) - 1\right) R(K) \left(\cos(\vartheta) + 1\right) \Big(-\frac{1}{2} R(K) + m\Big)\Big)\Big)$$

$$\frac{1}{-R(K) + 2m} \frac{1}{R(K)} = -4 \Big(W\Big(-\frac{1}{2} u v \mathrm{e}^{-1} \frac{1}{m}\Big) + 1\Big)^2 m^2 \sin(\vartheta)^2, \Big($$

$$-4 \Big(\frac{\partial}{\partial v} T(K)\Big) \Big(-\frac{1}{2} R(K) + m\Big)^2 \Big(\frac{\partial}{\partial u} T(K)\Big) + \Big(\frac{\partial}{\partial v} R(K)\Big) \Big(\frac{\partial}{\partial u}$$

$$R(K)\Big) R(K)^2\Big) \frac{1}{-R(K) + 2m} \frac{1}{R(K)} = -8 W\Big($$

$$-\frac{1}{2} u v \mathrm{e}^{-1} \frac{1}{m}\Big) m^2 \frac{1}{W\Big(-\frac{1}{2} u v \mathrm{e}^{-1} \frac{1}{m}\Big) + 1} \frac{1}{u} \frac{1}{v}, \Big(-4 \Big(\frac{\partial}{\partial v}$$

$$T(K)\Big) \Big(-\frac{1}{2} R(K) + m\Big)^2 \Big(\frac{\partial}{\partial \varphi} T(K)\Big) + \Big(\frac{\partial}{\partial v} R(K)\Big) \Big(\frac{\partial}{\partial \varphi} R(K)\Big) R(K)^2\Big)$$

$$\frac{1}{-R(K) + 2m} \frac{1}{R(K)} = 0, \Big(-4 \Big(\frac{\partial}{\partial v} T(K)\Big) \Big(-\frac{1}{2} R(K) + m\Big)^2 \Big(\frac{\partial}{\partial \vartheta}$$

$$T(K)\Big) + \Big(\frac{\partial}{\partial v} R(K)\Big) \Big(\frac{\partial}{\partial \vartheta} R(K)\Big) R(K)^2\Big) \frac{1}{-R(K) + 2m} \frac{1}{R(K)} = 0, \Big($$

$$-4 \Big(\frac{\partial}{\partial \varphi} T(K)\Big) \Big(-\frac{1}{2} R(K) + m\Big)^2 \Big(\frac{\partial}{\partial u} T(K)\Big) + \Big(\frac{\partial}{\partial \varphi} R(K)\Big) \Big(\frac{\partial}{\partial u}$$

$$R(K)\Big) R(K)^2\Big) \frac{1}{-R(K) + 2m} \frac{1}{R(K)} = 0, \Big(-4 \Big(\frac{\partial}{\partial \varphi} T(K)\Big) \Big($$

$$-\frac{1}{2} R(K) + m\Big)^2 \Big(\frac{\partial}{\partial \vartheta} T(K)\Big) + \Big(\frac{\partial}{\partial \varphi} R(K)\Big) \Big(\frac{\partial}{\partial \vartheta} R(K)\Big) R(K)^2\Big)$$

$$\frac{1}{-R(K) + 2m} \frac{1}{R(K)} = 0, \Big(-4 \Big(\frac{\partial}{\partial \vartheta} T(K)\Big) \Big(-\frac{1}{2} R(K) + m\Big)^2 \Big(\frac{\partial}{\partial u}$$

$$T(K)\Big) + \Big(\frac{\partial}{\partial \vartheta} R(K)\Big) \Big(\frac{\partial}{\partial u} R(K)\Big) R(K)^2\Big) \frac{1}{-R(K) + 2m} \frac{1}{R(K)} = 0,$$

$$-\Big(\frac{\partial}{\partial \vartheta} R(K)\Big)^2 R(K) \frac{1}{R(K) - 2m} - R(K)^2 + \Big(\frac{\partial}{\partial \vartheta} T(K)\Big)^2 (R(K)$$

$$-2m) \frac{1}{R(K)} = -4 \Big(W\Big(-\frac{1}{2} u v \mathrm{e}^{-1} \frac{1}{m}\Big) + 1\Big)^2 m^2 \Big\}:$$

> *declare*(*sys*)

$R(u, \vartheta, \varphi, v)$ *will now be displayed as* R

$T(u, \vartheta, \varphi, v)$ *will now be displayed as* T **(74)**

This is a complicated and unsimplifiable system, highly nonlinear and involving special functions

> *nops*(*sys*)

10 **(75)**

> simplify(sys)

$$\left\{ \frac{-4\left(-\frac{R}{2}+m\right)^2 T_u^{\,2} + R_u^{\,2} R^2}{(-R+2\,m)\,R} = 0,\ \frac{-4\left(-\frac{R}{2}+m\right)^2 T_v^{\,2} + R_v^{\,2} R^2}{(-R+2\,m)\,R} = 0, \right. \tag{76}$$

$$\frac{1}{(-R+2\,m)\,R}\left(-4\left(-\frac{R}{2}+m\right)^2 T_\varphi^{\,2} + 2\,R^2\left(\frac{R_\varphi^{\,2}}{2} + (\cos(\vartheta)\right.\right.$$

$$\left.\left.-1)\,R\,(\cos(\vartheta)+1)\left(-\frac{R}{2}+m\right)\right)\right) = -4\left(W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2\,m}\right)+1\right)^2 m^2 \sin(\vartheta)^2,$$

$$\frac{-4\,T_v\left(-\frac{R}{2}+m\right)^2 T_u + R_v\,R_u\,R^2}{(-R+2\,m)\,R} = -\frac{8\,W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2\,m}\right)m^2}{\left(W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2\,m}\right)+1\right)u\,v},$$

$$\frac{-4\,T_v\left(-\frac{R}{2}+m\right)^2 T_\varphi + R_v\,R_\varphi\,R^2}{(-R+2\,m)\,R} = 0,\ \frac{-4\,T_v\left(-\frac{R}{2}+m\right)^2 T_\vartheta + R_v\,R_\vartheta\,R^2}{(-R+2\,m)\,R} = 0,$$

$$\frac{-4\,T_\varphi\left(-\frac{R}{2}+m\right)^2 T_u + R_\varphi\,R_u\,R^2}{(-R+2\,m)\,R} = 0,\ \frac{-4\,T_\varphi\left(-\frac{R}{2}+m\right)^2 T_\vartheta + R_\varphi\,R_\vartheta\,R^2}{(-R+2\,m)\,R}$$

$$= 0,\ \frac{-4\,T_\vartheta\left(-\frac{R}{2}+m\right)^2 T_u + R_\vartheta\,R_u\,R^2}{(-R+2\,m)\,R} = 0,\ -\frac{R_\vartheta^{\,2}\,R}{R-2\,m} - R^2 + \frac{T_\vartheta^{\,2}\,(R-2\,m)}{R}$$

$$= -4\left(W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2\,m}\right)+1\right)^2 m^2$$

> indets(sys, specfunc(:-diff))

$$\{R_u, R_v, R_\varphi, R_\vartheta, T_u, T_v, T_\varphi, T_\vartheta\} \tag{77}$$

> casesplit(sys, caseplot)

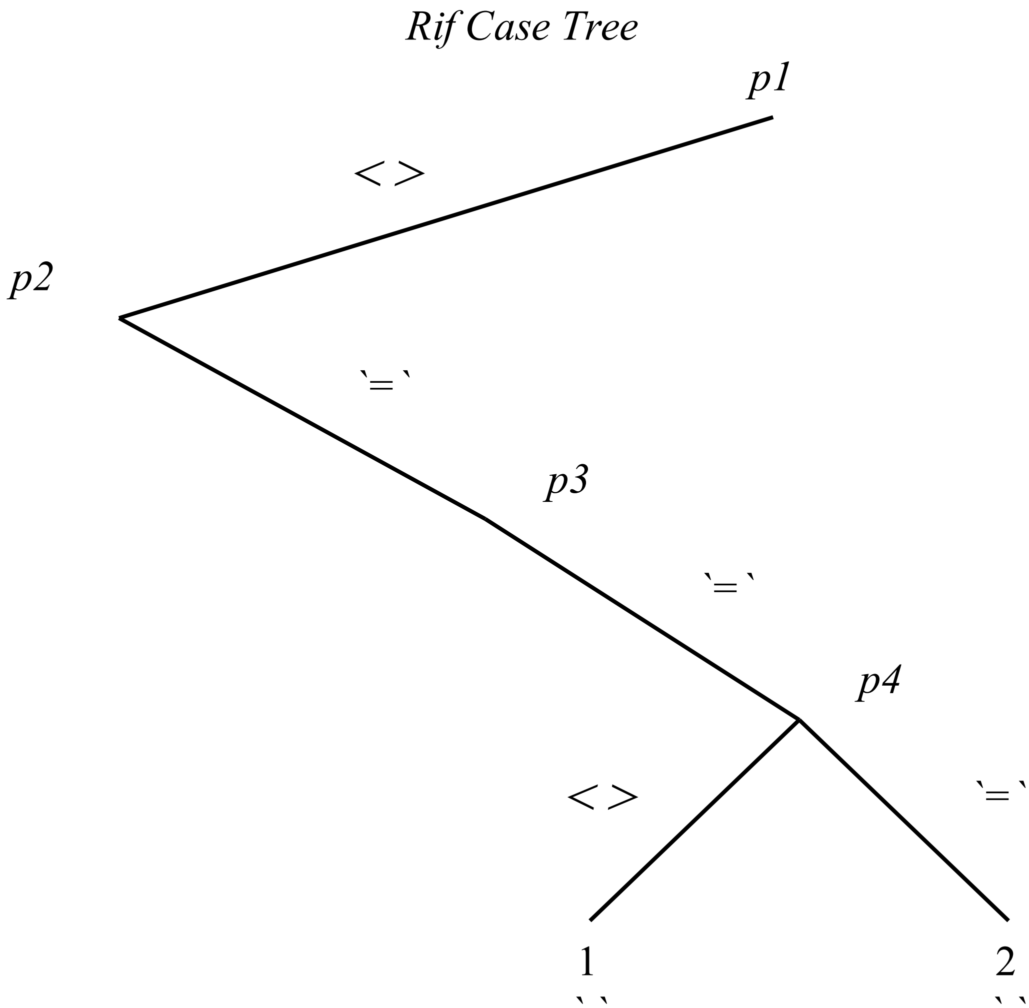
========= *Pivots Legend* =========

$$p1 = T_v$$

$$p2 = \_F2(\vartheta)^2 + \_F3(\vartheta)^2 - 1$$

$$p3 = -R^2 + 4\,m^2\,(\_F4(u,v)+1)^2$$

$$p4 = -T_v\,v + 2\,m$$



$$\left[T_u = \frac{2\,m}{u}, T_\vartheta = 0, T_\varphi = 0, T_v = -\frac{2\,m}{v}, R = 2\left(W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2\,m}\right)+1\right)m\right] \&\text{where } [\,], \left[T_u = -\frac{2\,m}{u}, T_\vartheta = 0, T_\varphi = 0, T_v = \frac{2\,m}{v}, R = 2\left(W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2\,m}\right)+1\right)m\right] \&\text{where } [\,] \tag{78}$$

> *pdsolve*(*sys*)

$$\left\{R = 2\left(W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2\,m}\right)+1\right)m, T = -2\,m\ln(u) + 2\,m\ln(v) + \_C1\right\}, \left\{R = 2\left(W\left(-\frac{u\,v\,\mathrm{e}^{-1}}{2\,m}\right)+1\right)m, T = 2\,m\ln(u) - 2\,m\ln(v) + \_C1\right\} \tag{79}$$

> *map*(*pdetest*, [%], *sys*)

$$[\{0\}, \{0\}] \tag{80}$$

>

Differential Algebra with anticommutative variables

How it works

> $restart;\ with(PDEtools),\ with(Physics):$

Set first θ and Q as suffixes for variables of type/anticommutative (see Physics[Setup])

> $Setup(anticommutativepre = \{Q, \theta\})$

** Partial match of 'anticommutativepre' against keyword 'anticommutativeprefix'*

$$[anticommutativeprefix = \{Q, \_\lambda, \theta\}] \tag{81}$$

Consider this anticommutative function $Q(x, y, \theta)$, of commutative and anticommutative variables

> $Q(x, y, \theta)$

$$Q(x, y, \theta) \tag{82}$$

It can always be expanded in power series of the anticommutative variable θ

> $ToFieldComponents(\%)$

$$\_Q1(x, y) + \_F1(x, y)\,\theta \tag{83}$$

> $ToSuperfields(\%)$

$$Q(x, y, \theta) \tag{84}$$

Consider this partial differential equation

> $diff(Q(x, y, \mathrm{theta}), x, \mathrm{theta})$

$$\frac{\partial^2}{\partial x \partial \theta} Q(x, y, \theta) \tag{85}$$

> $ToFieldComponents(\%)$

$$\frac{\partial}{\partial x} \_F1(x, y) \tag{86}$$

Its solution using pdsolve

> $pdsolve((\mathbf{85}))$

$$Q(x, y, \theta) = \_F2(x, y)\,\_\lambda1 + \_F4(y)\,\theta \tag{87}$$

> $diff(\%, \mathrm{theta}, x)$

$$\frac{\partial^2}{\partial x \partial \theta} Q(x, y, \theta) = 0 \tag{88}$$

Note the introduction of an anticommutative constant $\_\lambda1$, analogous to the commutative constants $\_Cn$ where n is an integer. The arbitrary functions $\_Fn$ introduced are all commutative as usual and the Grassmannian parity (on right-hand-side if compared with the one on the left-hand-side) is preserved

> $Physics\text{:-}GrassmannParity((\mathbf{87}))$

$$1 = 1 \tag{89}$$

So: "*To do differential elimination in systems with anticommutative variables and functions we need to 1) expand in series w.r.t the anticommutative variables, use a noncommutative product operator to carefully manipulate products of anticommutative functions*"

Example: PDE system with one unknown $Q(x, y, \theta_1, \theta_2)$

To avoid redundant typing in the input that follows and redundant display of information on the screen, use PDEtools:-declare, and PDEtools:-diff_table, that also handles anticommutative variables by automatically using Physics:-diff when Physics is loaded

> $PDEtools{:}\text{-}declare\left(Q\left(x, y, \theta_1, \theta_2\right)\right)$

$$Q\left(x, y, \theta_1, \theta_2\right) \textit{ will now be displayed as } Q \tag{90}$$

> $q := PDEtools{:}\text{-}diff\_table\left(Q\left(x, y, \theta_1, \theta_2\right)\right)$:

Now we can enter derivatives directly as the function's name indexed by the differentiation variables and see the display the same way; two PDEs

> $pde_1 := q_{x, y, \theta_1} + q_{x, y, \theta_2} - q_{y, \theta_1, \theta_2} = 0$

$$pde_1 := Q_{x, y, \theta_1} + Q_{x, y, \theta_2} - Q_{y, \theta_1, \theta_2} = 0 \tag{91}$$

> $pde_2 := q_{\theta_1} = 0$

$$pde_2 := Q_{\theta_1} = 0 \tag{92}$$

The solution to this system:

> $sys := \left[pde_1, pde_2\right]$

$$sys := \left[Q_{x, y, \theta_1} + Q_{x, y, \theta_2} - Q_{y, \theta_1, \theta_2} = 0, Q_{\theta_1} = 0\right] \tag{93}$$

> $pdsolve(sys)$

$$Q = \_F4(x, y)\ \_\lambda 2 + (\_F9(x) + \_F8(y))\ \theta_2 \tag{94}$$

How do we get there?

The derivatives in pde[2] can be substituted in pde[1] reducing the problem to a simpler one:

> $dsubs\left(pde_2, pde_1\right)$

$$Q_{x, y, \theta_2} = 0 \tag{95}$$

This one is solved as explained before

> $pdsolve(\mathbf{(95)})$

$$Q = \_F4(x, y)\ \_\lambda 2 + \_F5(x, y)\ \theta_1 + (\_F9(x) + \_F8(y))\ \theta_2 + (\_F11(x) + \_F10(y))\ \_\lambda 3\ \theta_1\ \theta_2 \tag{96}$$

Substituting this result for Q back into pde[2], then multiplying by θ_1 and subtracting from the above leads to the PDE system solution returned by pdsolve.

Using differential elimination techniques pde[2] can be "reduced" using pde[1]

> $PDEtools{:}\text{-}ReducedForm\left(pde_1, pde_2\right)$

$$\left[Q_{x, y, \theta_2}\right] \text{ \&where } [\ ] \tag{97}$$

Or: *simplify the system taking the integrability conditions into account:*

> $casesplit([pde_1, pde_2])$

$$\left[Q_{x, y, \theta_2} = 0, Q_{\theta_1} = 0\right] \&\text{where } [\ ] \tag{98}$$

>

Example: The Lie symmetry infinitesimals of the PDE system for $Q(x, y, \theta_1, \theta_2)$

Set fthe generic form of the infinitesimals for a PDE system like this one formed by pde[1] and pde[2].
We need anticommutative infinitesimals for the dependent anticommutative function Q and two of the independent anticommutative variables θ_1, θ_2. We use $\Xi(x, y, \theta_1, \theta_2)$ and $\mathrm{H}(x, y, \theta_1, \theta_2)$ for the anticommutative infinitesimal symmetry generators and the corresponding lower case greek letters for commutative ones

> $Setup(anticommutativepre = \{\mathrm{H}, \Xi\}, additionally)$

** Partial match of 'anticommutativepre' against keyword 'anticommutativeprefix'*

$$[anticommutativeprefix = \{\mathrm{H}, Q, \Xi, \_Q, \_\lambda, \theta\}] \tag{99}$$

> $S := [\xi_1, \xi_2, \Xi_1, \Xi_2, \mathrm{H}](x, y, \theta_1, \theta_2)$

$$S := [\xi_1(x, y, \theta_1, \theta_2), \xi_2(x, y, \theta_1, \theta_2), \Xi_1(x, y, \theta_1, \theta_2), \Xi_2(x, y, \theta_1, \theta_2), \mathrm{H}(x, y, \theta_1, \theta_2)] \tag{100}$$

> $PDEtools{:-}declare(S)$

$\mathrm{H}(x, y, \theta_1, \theta_2)$ will now be displayed as H

$\Xi(x, y, \theta_1, \theta_2)$ will now be displayed as Ξ

$\xi(x, y, \theta_1, \theta_2)$ will now be displayed as ξ **(101)**

The corresponding InfinitesimalGenerator

> $InfinitesimalGenerator(S, Q(x, y, \theta_1, \theta_2))$

$$f \mapsto \xi_1(x, y, \theta_1, \theta_2)\left(\frac{\partial}{\partial x} f\right) + \xi_2(x, y, \theta_1, \theta_2)\left(\frac{\partial}{\partial y} f\right) + \Xi_1(x, y, \theta_1, \theta_2)\left(\frac{\partial}{\partial \theta_1} f\right) + \Xi_2(x, y, \theta_1, \theta_2)\left(\frac{\partial}{\partial \theta_2} f\right) + \mathrm{H}(x, y, \theta_1, \theta_2)\left(\frac{\partial}{\partial Q} f\right) \tag{102}$$

The prolongation of the infinitesimal for Q is computed with Eta_k, assign it here to the lower case η to use more familiar notation (recall $q[\ ] = Q(x, y, \theta_1, \theta_2)$)

> $\eta := Eta\_k(S, q[\ ])$

$$\eta := \eta \tag{103}$$

The first prolongations of η with respect to x and θ_1

> $\eta_{Q, [x]}$

$$H_x - (\xi_1)_x Q_x - (\xi_2)_x Q_y - Q_{\theta_1} (\Xi_1)_x - Q_{\theta_2} (\Xi_2)_x \quad (104)$$

> $\eta_{Q, [\theta_1]}$

$$H_{\theta_1} + Q_x (\xi_1)_{\theta_1} + Q_y (\xi_2)_{\theta_1} - Q_{\theta_1} (\Xi_1)_{\theta_1} - Q_{\theta_2} (\Xi_2)_{\theta_1} \quad (105)$$

The second mixed prolongations of η with respect to x, y and x, θ_1

> $\eta_{Q, [x, y]}$

$$H_{x, y} - (\xi_1)_{x, y} Q_x - (\xi_2)_{x, y} Q_y - Q_{\theta_1} (\Xi_1)_{x, y} - Q_{\theta_2} (\Xi_2)_{x, y} - (\xi_1)_y Q_{x, x} - (\xi_2)_y Q_{x, y} - Q_{x, \theta_1} (\Xi_1)_y - Q_{x, \theta_2} (\Xi_2)_y - (\xi_1)_x Q_{x, y} - (\xi_2)_x Q_{y, y} - Q_{y, \theta_1} (\Xi_1)_x - Q_{y, \theta_2} (\Xi_2)_x \quad (106)$$

> $\eta_{Q, [x, \theta_1]}$

$$H_{x, \theta_1} + Q_x (\xi_1)_{x, \theta_1} + Q_y (\xi_2)_{x, \theta_1} - Q_{\theta_1} (\Xi_1)_{x, \theta_1} - Q_{\theta_2} (\Xi_2)_{x, \theta_1} + Q_{x, x} (\xi_1)_{\theta_1} + Q_{x, y} (\xi_2)_{\theta_1} - Q_{x, \theta_1} (\Xi_1)_{\theta_1} - Q_{x, \theta_2} (\Xi_2)_{\theta_1} - Q_{x, \theta_1} (\xi_1)_x - Q_{y, \theta_1} (\xi_2)_x - Q_{\theta_1, \theta_2} (\Xi_2)_x \quad (107)$$

The DeterminingPDE for this system

> $DeterminingPDE([pde_1, pde_2], S)$

$$\Big\{ -(\xi_2)_{\theta_1} = 0, -(\xi_2)_{\theta_2} = 0, -(\xi_1)_{\theta_1, \theta_2} \theta_1 - (\xi_1)_{\theta_2} = 0, -(\xi_1)_{\theta_1} + (\xi_1)_{\theta_1, \theta_2} \theta_2 = 0, -\theta_1 (\xi_1)_{y, \theta_1} - \theta_2 (\xi_1)_{y, \theta_2} + (\xi_1)_y = 0, -\theta_1 (\xi_1)_{x, x, \theta_1} - \theta_2 (\xi_1)_{x, x, \theta_2} + (\xi_1)_{x, x} = 0, -\theta_1 (\xi_2)_{x, \theta_1} - \theta_2 (\xi_2)_{x, \theta_2} + (\xi_2)_x = 0, (\Xi_1)_x - (\Xi_1)_{x, \theta_2} \theta_2 - (\Xi_1)_{x, \theta_1} \theta_1 = 0, (\Xi_2)_x - (\Xi_2)_{x, \theta_2} \theta_2 - (\Xi_2)_{x, \theta_1} \theta_1 = 0, (\xi_1)_{\theta_1, \theta_2} = 0, (\xi_2)_{\theta_1, \theta_2} = 0, (\Xi_1)_{\theta_1, \theta_2} = 0, (\Xi_1)_{\theta_1} = -\theta_1 (\xi_1)_{x, \theta_1} - \theta_2 (\xi_1)_{x, \theta_2} + (\xi_1)_x, (\Xi_1)_{\theta_2} = (\Xi_2)_{\theta_2} + \theta_1 (\xi_1)_{x, \theta_1} + \theta_2 (\xi_1)_{x, \theta_2} - (\xi_1)_x, (\Xi_2)_{x, \theta_2} = 0, (\Xi_2)_{y, \theta_2} = 0, (\Xi_2)_{\theta_1, \theta_2} = 0, (\Xi_2)_{\theta_1} = 0, H_{\theta_1, \theta_2} = 0, H_{x, y, \theta_2} = 0, H_{\theta_1} = 0 \quad (108)$$

To compute now the exact form of the symmetry infinitesimals you can either solve this PDE system for the commutative and anticommutative functions using pdsolve, or directly pass the system to Infinitesimals that will perform all these steps automatically

> $Infinitesimals([pde_1, pde_2], q[\,], S)$

$$[1, \_F3(y), \_F1(y)\,\_\lambda 6, \_F2(y)\,\_\lambda 8, \_F6(x, y)\,\_\lambda 4 + (\_F5(x) + \_F4(y))\,\theta_2], [x, \_F9(y), \_F7(y)\,\_\lambda 6 + \theta_1 - \theta_2, \_F8(y)\,\_\lambda 8, \_F12(x, y)\,\_\lambda 4 + (\_F11(x) + \_F10(y))\,\theta_2], [0, \_F15(y), \_F13(y)\,\_\lambda 6 + \theta_2, \_F14(y)\,\_\lambda 8 + \theta_2, \_F18(x, y)\,\_\lambda 4 + (\_F17(x) + \_F16(y))\,\theta_2] \quad \textbf{(109)}$$

To see these three list of symmetry infinitesimals with a label in the left-hand-side, you can use map

> **for** EE **in** $map_3(zip, \grave{}=\grave{}, S, [\textbf{(109)}])$ **do** EE **od**;

$$[\xi_1 = 1, \xi_2 = \_F3(y), \Xi_1 = \_F1(y)\,\_\lambda 6, \Xi_2 = \_F2(y)\,\_\lambda 8, \mathrm{H} = \_F6(x, y)\,\_\lambda 4 + (\_F5(x) + \_F4(y))\,\theta_2$$

$$[\xi_1 = x, \xi_2 = \_F9(y), \Xi_1 = \_F7(y)\,\_\lambda 6 + \theta_1 - \theta_2, \Xi_2 = \_F8(y)\,\_\lambda 8, \mathrm{H} = \_F12(x, y)\,\_\lambda 4 + (\_F11(x) + \_F10(y))\,\theta_2$$

$$[\xi_1 = 0, \xi_2 = \_F15(y), \Xi_1 = \_F13(y)\,\_\lambda 6 + \theta_2, \Xi_2 = \_F14(y)\,\_\lambda 8 + \theta_2, \mathrm{H} = \_F18(x, y)\,\_\lambda 4 + (\_F17(x) + \_F16(y))\,\theta_2 \quad \textbf{(110)}$$

To verify this result we use SymmetryTest - it also handles anticommutative variables

> $map(SymmetryTest, [\textbf{(109)}], [pde_1, pde_2])$

$$[\{0\}, \{0\}, \{0\}] \quad \textbf{(111)}$$

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