The FunctionAdvisor: extending information on mathematical functions

with computer algebra algorithms

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Abstract:
A shift in paradigm is happening, from: encoding information into a database, to: encoding essential blocks of information together with algorithms within a computer algebra system. Then, the information is not only searchable but can also be recreated in many different ways and actually used to compute. This talk focuses on this shift in paradigm over a real case example: the digitizing of information regarding mathematical functions as the FunctionAdvisor project of the Maple computer algebra system.

The FunctionAdvisor (basic)

> *restart*; *FunctionAdvisor*( )

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The usage is as follows:
    > FunctionAdvisor( topic, function, ... );
where 'topic' indicates the subject on which advice is required,
'function' is the name of a Maple function, and '...' represents
possible additional input depending on the 'topic' chosen. To
list the possible topics:
    > FunctionAdvisor( topics );
A short form usage,
    > FunctionAdvisor( function );
with just the name of the function is also available and
displays a summary of information about the function.
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> *FunctionAdvisor*(*topics*)

```
The topics on which information is available are:
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[*DE*, *analytic_extension*, *asymptotic_expansion*, *branch_cuts*, *branch_points*, *calling_sequence*, *class_members*, *classify_function*, *definition*, *describe*, *differentiation_rule*, *function_classes*, *identities*, *integral_form*, *known_functions*, *periodicity*, *plot*, *relate*, *required_assumptions*, *series*, *singularities*, *special_values*, *specialize*, *sum_form*, *symmetries*, *synonyms*, *table*] **(1.1)**

> *FunctionAdvisor*(*function*, *quiet*)

[*trig*, *trigh*, *arctrig*, *arctrigh*, *elementary*, *GAMMA_related*, *Psi_related*, *Kelvin*, *Airy*, *Hankel*, *Bessel_related*, *0F1*, *orthogonal_polynomials*, *Ei_related*, *erf_related*, *Kummer*, *Whittaker*, *Cylinder*, *1F1*, *Elliptic_related*, *Legendre*, *Chebyshev*, *2F1*, *Lommel*, *Struve_related*, *hypergeometric*, *Jacobi_related*, *InverseJacobi_related*, *Elliptic_doubly_periodic*, *Weierstrass_related*, *Zeta_related*, *complex_components*, *piecewise_related*, *Other*, *Bell*, *Heun*, *Appell*, *trigall*, *arctrigall*, *integral_transforms*] **(1.2)**

> *FunctionAdvisor*(*bess*)

```
* Partial match of "bess" against topic "Bessel_related".
The 14 functions in the "Bessel_related" class are:
```

$$[\mathrm{AiryAi}, \mathrm{AiryBi}, \mathrm{BesselI}, \mathrm{BesselJ}, \mathrm{BesselK}, \mathrm{BesselY}, \mathrm{HankelH1}, \mathrm{HankelH2}, \mathrm{KelvinBei}, \mathrm{KelvinBer}, \mathrm{KelvinHei}, \mathrm{KelvinHer}, \mathrm{KelvinKei}, \mathrm{KelvinKer}] \tag{1.3}$$

> *FunctionAdvisor*(*describe*, BesselK)

$$\mathrm{BesselK} = \textit{Modified Bessel function of the second kind} \tag{1.4}$$

> *FunctionAdvisor*(*sum*, tan)

```
* Partial match of "sum" against topic "sum_form".
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$$\left[\tan(z) = \sum_{\_kI=1}^{\infty} \frac{B_{2\_kI}\,(-1)^{\_kI}\, z^{-1+2\_kI}\,\left(4^{\_kI} - 16^{\_kI}\right)}{\Gamma(2\_kI+1)}, |z| < \frac{\pi}{2}\right] \tag{1.5}$$

> *FunctionAdvisor*(*integral*, Beta)

```
* Partial match of "integral" against topic "integral_form".
```

$$\left[B(x, y) = \int_0^1 \_kI^{x-1}\,(1-\_kI)^{y-1}\, \mathrm{d}\_kI, 0 < \Re(x) \wedge 0 < \Re(y)\right] \tag{1.6}$$

More complicated relationships between mathematical functions, computed using Maple internal database and algorithms.

> *FunctionAdvisor*(*specialize*, HermiteH, KummerU)

$$\left[H_a(z) = 2^a\, U\left(-\frac{a}{2}, \frac{1}{2}, z^2\right), 0 < \Re(z) \vee (\Re(z) = 0 \wedge 0 < \Im(z))\right] \tag{1.7}$$

> *FunctionAdvisor*(*DE*, EllipticF(z, k))

$$\left[f(z,k) = F(z,k), \left[\frac{\partial^2}{\partial k^2} f(z,k) = \frac{\left(-1 - 3k^4 z^2 + (z^2+3)k^2\right)\left(\frac{\partial}{\partial k} f(z,k)\right)}{k^5 z^2 + (-z^2-1)k^3 + k} + \frac{(z^3 - z)\left(\frac{\partial}{\partial z} f(z,k)\right)}{(k^4-k^2)z^2 - k^2 + 1} + \frac{(-k^2 z^2 + 1) f(z,k)}{k^4 z^2 - k^2 z^2 - k^2 + 1}, \frac{\partial^2}{\partial k \partial z} f(z,k) = -\frac{\left(\frac{\partial}{\partial z} f(z,k)\right) k z^2}{k^2 z^2 - 1}, \frac{\partial^2}{\partial z^2} f(z,k) = \frac{(-2k^2 z^3 + (k^2+1)z)\left(\frac{\partial}{\partial z} f(z,k)\right)}{1 + k^2 z^4 + (-k^2-1)z^2}\right]\right] \tag{1.8}$$

The information returned by the FunctionAdvisor command can be used for further computations: verify the above

> *pdetest*(*op*(%))

$$[0, 0, 0] \tag{1.9}$$

The relation between all elementary functions and the pFq hypergeometric function:

> *FunctionAdvisor*(*elementary*, *quiet*)

$$[\arccos, \mathrm{arccosh}, \mathrm{arccot}, \mathrm{arccoth}, \mathrm{arccsc}, \mathrm{arccsch}, \mathrm{arcsec}, \mathrm{arcsech}, \arcsin, \mathrm{arcsinh}, \arctan, \mathrm{arctanh}, \cos, \cosh, \cot, \coth, \csc, \mathrm{csch}, \exp, \ln, \sec, \mathrm{sech}, \sin, \sinh, \tan, \tanh] \tag{1.10}$$

> *map2*(*FunctionAdvisor*, *relate*, **(1.10)**, hypergeom)

$$\arccos(z) = \frac{\pi}{2} - z\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right), \operatorname{arccosh}(z) \tag{1.11}$$

$$= \frac{\sqrt{-(-1+z)^2}\left(2 z\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) - \pi\right)}{-2+2z}, \operatorname{arccot}(z) = \frac{\pi}{2} - z\,{}_2F_1\left(\frac{1}{2}, 1;\right.$$

$$\left.\frac{3}{2}; -z^2\right), \operatorname{arccoth}(z) = \frac{\pi\sqrt{-(z-1)^2} + 2 z\,{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right)(z-1)}{2z-2}, \operatorname{arccsc}(z)$$

$$= \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1}{z^2}\right)}{z}, \operatorname{arccsch}(z) = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -\frac{1}{z^2}\right)}{z}, \operatorname{arcsec}(z) = \frac{\pi}{2}$$

$$- \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1}{z^2}\right)}{z}, \operatorname{arcsech}(z)$$

$$= \frac{\sqrt{-\frac{(z-1)^2}{z^2}}\left(z\pi - 2\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1}{z^2}\right)\right)}{2z-2}, \arcsin(z) = z\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2};\right.$$

$$\left.z^2\right), \operatorname{arcsinh}(z) = z\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2\right), \arctan(z) = z\,{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right),$$

$$\operatorname{arctanh}(z) = z\,{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right), \cos(z) = {}_0F_1\left(; \frac{1}{2}; -\frac{z^2}{4}\right), \cosh(z) = {}_0F_1\left(; \frac{1}{2};\right.$$

$$\left.\frac{z^2}{4}\right), \cot(z) = \frac{{}_0F_1\left(;\frac{1}{2};-\frac{z^2}{4}\right)}{z\,{}_0F_1\left(;\frac{3}{2};-\frac{z^2}{4}\right)}, \coth(z) = \frac{{}_0F_1\left(;\frac{1}{2};\frac{z^2}{4}\right)}{z\,{}_0F_1\left(;\frac{3}{2};\frac{z^2}{4}\right)}, \csc(z)$$

$$= \frac{1}{z\,{}_0F_1\left(;\frac{3}{2};-\frac{z^2}{4}\right)}, \operatorname{csch}(z) = \frac{1}{z\,{}_0F_1\left(;\frac{3}{2};\frac{z^2}{4}\right)}, \mathrm{e}^z = {}_0F_0(;;z), \ln(z) = (z-1)$$

$${}_2F_1(1,1;2;-z+1), \sec(z) = \frac{1}{{}_0F_1\left(;\frac{1}{2};-\frac{z^2}{4}\right)}, \operatorname{sech}(z) = \frac{1}{{}_0F_1\left(;\frac{1}{2};\frac{z^2}{4}\right)}, \sin(z)$$

$$= z\,{}_0F_1\left(;\frac{3}{2};-\frac{z^2}{4}\right), \sinh(z) = z\,{}_0F_1\left(;\frac{3}{2};\frac{z^2}{4}\right), \tan(z) = \frac{z\,{}_0F_1\left(;\frac{3}{2};-\frac{z^2}{4}\right)}{{}_0F_1\left(;\frac{1}{2};-\frac{z^2}{4}\right)},$$

$$\left.\tanh(z) = \frac{z\,{}_0F_1\left(;\frac{3}{2};\frac{z^2}{4}\right)}{{}_0F_1\left(;\frac{1}{2};\frac{z^2}{4}\right)}\right]$$

All the 'specializations' of the arcsin function

> *FunctionAdvisor*(*specialize*, arcsin)

$$\left[\arcsin(z) = z\,F_1\left(\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, 0, z^2\right), \textit{with no restrictions on } (z)\right], \left[\arcsin(z)\right. \tag{1.12}$$

$$\left.= z\,F_2\left(\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 0, z^2\right), \textit{with no restrictions on } (z)\right], \left[\arcsin(z) = z\,F_3\left(0, \frac{1}{2},\right.\right.$$

$$\left.\left.0, \frac{1}{2}, \frac{3}{2}, 0, z^2\right), \textit{with no restrictions on } (z)\right], \left[\arcsin(z) = z\,F_4\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 0, z^2\right),\right.$$

$$\left.\textit{with no restrictions on } (z)\right], \left[\arcsin(z) = \frac{z\,HC\left(0, \frac{1}{2}, 0, 0, \frac{1}{4}, \frac{z^2}{z^2-1}\right)}{\sqrt{-z^2+1}},\right.$$

$$\left.\textit{with no restrictions on } (z)\right], \left[\arcsin(z) = z\,HG\left(0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, z^2\right),\right.$$

$$\left.\textit{with no restrictions on } (z)\right], \left[\arcsin(z) = \frac{\pi}{2} + \frac{am^{-1}(\operatorname{arcsec}(z)|1)\,(z-1)}{\sqrt{-(z-1)^2}}, \Re(z)\right.$$

$$\in\ (0, \pi)\Big],\ \left[\arcsin(z) = \frac{z\,\pi\,P_{-\frac{1}{2}}^{\left(\frac{1}{2},\,-\frac{1}{2}\right)}\left(-2 z^2 + 1\right)}{2},\ \textit{with no restrictions on } (z)\right],$$

$$\left[\arcsin(z) = \frac{z \sqrt{\pi}\,\left(-2 z^2 + 2\right)^{1/4} P_{-\frac{1}{2}}^{-\frac{1}{2}}\left(-2 z^2 + 1\right)}{2\left(-2 z^2\right)^{1/4}},\ \textit{with no restrictions on } (z)\right],$$

$$\left[\arcsin(z) = \frac{z\, G_{2,2}^{1,2}\left(-z^2\ \middle|\ \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 0, -\frac{1}{2} \end{matrix}\right)}{2\sqrt{\pi}},\ \textit{with no restrictions on } (z)\right],\ \left[\arcsin(z) = \frac{\pi}{2} - \arccos(z),\ \textit{with no restrictions on } (z)\right],$$

$$\left[\arcsin(z) = \frac{\pi}{2} + \frac{\operatorname{arccosh}(z)\,(z-1)}{\sqrt{-(z-1)^2}},\ \textit{with no restrictions on } (z)\right],\ \left[\arcsin(z) = \pi - 2\operatorname{arccot}\left(\frac{z}{1 + \sqrt{-z^2 + 1}}\right),\ \textit{with no restrictions on } (z)\right],$$

$$\left[\arcsin(z) = \frac{1}{\mathrm{I}\, z + \sqrt{-z^2 + 1} + 1}\left(\left(2\,\mathrm{I}\sqrt{-z^2 + 1} + 2\,\mathrm{I} - 2 z\right)\operatorname{arccoth}\left(\frac{-\mathrm{I}\, z}{1 + \sqrt{-z^2 + 1}}\right) + \mathrm{I}\left(1 + \sqrt{-z^2 + 1}\right)\pi\sqrt{-\left(\frac{\mathrm{I}\, z}{1 + \sqrt{-z^2 + 1}} + 1\right)^2}\right),\ \textit{with no restrictions on } (z)\right],$$

$$\left[\arcsin(z) = \operatorname{arccsc}\left(\frac{1}{z}\right),\ \textit{with no restrictions on } (z)\right],\ \left[\arcsin(z) = \mathrm{I}\operatorname{arccsch}\left(\frac{\mathrm{I}}{z}\right),\ \textit{with no restrictions on } (z)\right],\ \left[\arcsin(z) = \frac{\pi}{2} - \operatorname{arcsec}\left(\frac{1}{z}\right),\ \textit{with no restrictions on } (z)\right],$$

$$\left[\arcsin(z) = \frac{\pi}{2} + \frac{\operatorname{arcsech}\left(\frac{1}{z}\right)(-1 + z)}{\sqrt{-\left(\frac{1}{z} - 1\right)^2 z^2}},\ \textit{with no restrictions on } (z)\right],\ \left[\arcsin(z) = -\mathrm{I}\operatorname{arcsinh}(\mathrm{I}\, z),\ \textit{with no restrictions on } (z)\right],$$

$\left[\arcsin(z) = 2\arctan\left(\frac{z}{1+\sqrt{-z^2+1}}\right), \textit{with no restrictions on } (z)\right], \left[\arcsin(z) = -2\,\mathrm{I}\,\mathrm{arctanh}\left(\frac{\mathrm{I}\,z}{1+\sqrt{-z^2+1}}\right), \textit{with no restrictions on } (z)\right], \left[\arcsin(z) = z\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right), \textit{with no restrictions on } (z)\right], \left[\arcsin(z) = -\mathrm{I}\ln\left(\mathrm{I}\,z+\sqrt{-z^2+1}\right), \textit{with no restrictions on } (z)\right]$

Summarize about a function

> *FunctionAdvisor*(GAMMA)

GAMMA

- **describe**
- **definition**
- **analytic extension**
- **classify function**
- **periodicity**
- **plot**
- **singularities**
- **branch points**
- **branch cuts**
- **special values**
- **identities**
- **sum form**
- **series**
- **asymptotic expansion**
- **integral form**
- **differentiation rule**
- **DE**

General description

• The FunctionAdvisor command provides information on the following topics.

analytic_extension	asymptotic_expansion	branch_cuts	branch_points
calling_sequence	class_members	classify_function	DE
definition	describe	differentiation_rule	display
function_classes	identities	integral_form	known_functions
relate	series	singularities	special_values

specialize	sum_form	synonyms	topics

The FunctionAdvisor command provides information on the following mathematical functions.

abs	AiryAi	AiryBi	AngerJ
arccos	arccosh	arccot	arccoth
arccsc	arccsch	arcsec	arcsech
arcsin	arcsinh	arctan	arctanh
argument	BellB	bernoulli	BesselI
BesselJ	BesselK	BesselY	Beta
binomial	ChebyshevT	ChebyshevU	Chi
Ci	CompleteBellB	cos	cosh
cot	coth	CoulombF	csc
csch	csgn	CylinderD	CylinderU
CylinderV	dawson	dilog	Dirac
doublefactorial	Ei	EllipticCE	EllipticCK
EllipticCPi	EllipticE	EllipticF	EllipticK
EllipticModulus	EllipticNome	EllipticPi	erf
erfc	erfi	euler	exp
factorial	FresnelC	Fresnelf	Fresnelg
FresnelS	GAMMA	GaussAGM	GegenbauerC
HankelH1	HankelH2	harmonic	Heaviside
HermiteH	HeunB	HeunBPrime	HeunC
HeunCPrime	HeunD	HeunDPrime	HeunG
HeunGPrime	HeunT	HeunTPrime	hypergeom
Hypergeom	Im	IncompleteBellB	InverseJacobiAM
InverseJacobiCD	InverseJacobiCN	InverseJacobiCS	InverseJacobiDC
InverseJacobiDN	InverseJacobiDS	InverseJacobiNC	InverseJacobiND
InverseJacobiNS	InverseJacobiSC	InverseJacobiSD	InverseJacobiSN
JacobiAM	JacobiCD	JacobiCN	JacobiCS

JacobiDC	JacobiDN	JacobiDS	JacobiNC
JacobiND	JacobiNS	JacobiP	JacobiSC
JacobiSD	JacobiSN	JacobiTheta1	JacobiTheta2
JacobiTheta3	JacobiTheta4	JacobiZeta	KelvinBei
KelvinBer	KelvinHei	KelvinHer	KelvinKei
KelvinKer	KummerM	KummerU	LaguerreL
LambertW	LegendreP	LegendreQ	LerchPhi
Li	ln	lnGAMMA	log
LommelS1	LommelS2	MathieuA	MathieuB
MathieuC	MathieuCE	MathieuCEPrime	MathieuCPrime
MathieuExponent	MathieuFloquet	MathieuFloquetPrime	MathieuS
MathieuSE	MathieuSEPrime	MathieuSPrime	MeijerG
pochhammer	polylog	Psi	Re
sec	sech	Shi	Si
signum	sin	sinh	SphericalY
Ssi	Stirling1	Stirling2	StruveH
StruveL	tan	tanh	WeberE
WeierstrassP	WeierstrassPPrime	WeierstrassSigma	WeierstrassZeta
WhittakerM	WhittakerW	Wrightomega	Zeta

Like the conversion facility for mathematical functions, the FunctionAdvisor command also works with the concept of function classes and considers assumptions on the function parameters, if any. The FunctionAdvisor command provides information on the following function classes.

\`0F1\`	\`1F1\`	\`2F1\`	Airy
arctrig	arctrigh	Bessel_related	Chebyshev
Cylinder	Ei_related	elementary	Elliptic_doubly_periodic
Elliptic_related	erf_related	GAMMA_related	Hankel
hypergeometric	Jacobi_related	Kelvin	Kummer
Legendre	Lommel	orthogonal_polynomials	Other

Psi_related Struve_related trig trigh

Weierstrass_related Whittaker Zeta_related

- The FunctionAdvisor command can be considered to be between a help and a computational special function facility. Due to the wide range of information this command can handle and in order to facilitate its use, it includes two distinctive features:
  - If you call the FunctionAdvisor command without arguments, it returns information that you can follow until the appropriate information displays.
  - If you call the FunctionAdvisor command with a topic or function misspelled, but a match exists, it returns the information with a warning message.

You do not have to remember the exact Maple name of each mathematical function or the FunctionAdvisor topic. However, to avoid these messages and all FunctionAdvisor verbosity, specify the optional argument quiet when calling the FunctionAdvisor command from another routine.

\>

Beyond the concept of a database

" Mathematical functions, are defined by algebraic expressions. So consider algebraic expressions in general ..."

Formal power series for algebraic expressions

This can go in a database

\> $restart;\ \sin(x) = convert(\sin(x), FormalPowerSeries)$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \textbf{(2.1.1)}$$

What follows builds on top of things like the above, but cannot go into a database: they are *generic* algebraic expressions

\> $\sqrt{\frac{\left(1-\sqrt{1-x}\right)}{x}}$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} \quad \textbf{(2.1.2)}$$

\> $\% = convert(\%, FormalPowerSeries)$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{k=0}^{\infty} \frac{\sqrt{2}\,(4k)!\,16^{-k} x^k}{2\,(2k)!^2\,(2k+1)} \quad \textbf{(2.1.3)}$$

\> $e^{\arcsin(x)}$

$$e^{\arcsin(x)} \quad \textbf{(2.1.4)}$$

\> $\% = convert(\%, FormalPowerSeries)$

$$e^{\arcsin(x)} = \left(\sum_{k=0}^{\infty} \frac{\left(\prod_{j=0}^{k} (4j^2+1)\right) x^{2k}}{(4k^2+1)\,(2k)!}\right) + \left(\sum_{k=0}^{\infty} \frac{\left(\prod_{j=0}^{k} (2j^2+2j+1)\right) 2^k x^{2k+1}}{(2k^2+2k+1)\,(2k+1)!}\right) \quad \textbf{(2.1.5)}$$

> $\frac{t}{(1 - x\,t - t^2)}$

$$\frac{t}{-t^2 - x\,t + 1} \quad \textbf{(2.1.6)}$$

> $\% = convert(\%, FormalPowerSeries, t)$

$$\frac{t}{-t^2 - x\,t + 1} = \sum_{k=0}^{\infty} \left(-\frac{\left(\frac{x}{2} - \frac{\sqrt{x^2+4}}{2}\right)^k}{\sqrt{x^2+4}} + \frac{\left(\frac{x}{2} + \frac{\sqrt{x^2+4}}{2}\right)^k}{\sqrt{x^2+4}} \right) t^k \quad \textbf{(2.1.7)}$$

> $\ln\left(\frac{(1+x^2)}{(1-x)}\right)$

$$\ln\left(\frac{x^2+1}{1-x}\right) \quad \textbf{(2.1.8)}$$

> $\% = convert(\%, FormalPowerSeries, method = rational)$

$$\ln\left(\frac{x^2+1}{1-x}\right) = \sum_{k=0}^{\infty} \frac{\left(\mathrm{I}\,(-\mathrm{I})^k - \mathrm{I}\,\mathrm{I}^k + (-\mathrm{I})^k\,\mathrm{I}^k\right) x^{k+1}}{(-\mathrm{I})^k\,\mathrm{I}^k\,(k+1)} \quad \textbf{(2.1.9)}$$

> $e^x \sin(x)$

$$e^x \sin(x) \quad \textbf{(2.1.10)}$$

> $\% = convert(\%, FormalPowerSeries, method = hypergeometric)$

$$e^x \sin(x) = \sum_{k=0}^{\infty} \left(-\frac{\mathrm{I}\,(1+\mathrm{I})^k}{2\,k!} + \frac{\mathrm{I}\,(1-\mathrm{I})^k}{2\,k!} \right) x^k \quad \textbf{(2.1.11)}$$

> $e^x - 2\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2} + \frac{\pi}{3}\right)$

$$e^x - 2\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2} + \frac{\pi}{3}\right) \quad \textbf{(2.1.12)}$$

> $\% = convert(\%, FormalPowerSeries, method = exponential)$

$$e^x - 2\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2} + \frac{\pi}{3}\right) = \sum_{k=0}^{\infty} \left(-\frac{\cos\left(\frac{2k\pi}{3}\right)}{k!} + \frac{\sqrt{3}\,\sin\left(\frac{2k\pi}{3}\right)}{k!} + \frac{1}{k!} \right) x^k \quad \textbf{(2.1.13)}$$

> $\%\% = convert(\%\%, FormalPowerSeries, method = hypergeometric)$

(2.1.14)

$$e^{x} - 2\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2} + \frac{\pi}{3}\right) = \sum_{k=0}^{\infty} \frac{3\,x^{3k+1}}{(3k+1)!} \tag{2.1.14}$$

References

- Koepf, Wolfram. "Algorithmic development of power series. Artificial intelligence and symbolic mathematical computing." **Lecture Notes in Computer Science**, Vol. **737**, pp. 195 -213. Edited by J. Calmet and J. A. Campbell. Berlin-Heidelberg: Springer, 1993.

\>

Differential polynomial forms for algebraic expressions

Extending the mathematical language to include the inverse functions

Consider the inverse of the functions of the mathematical language, for example:

\> *with*(*PDEtools*) : *declare*($y(x)$, *prime* = x);

$y(x)$ will now be displayed as y

derivatives with respect to x of functions of one variable will now be displayed with ' **(2.2.1.1)**

\> $y(x) = \mathrm{dawson}^{(-1)}(x)$

$$y = \mathrm{dawson}^{(-1)}(x) \tag{2.2.1.2}$$

\> *dpolyform*(**(2.2.1.2)**, *no_Fn*)

$$\left[y' = -\frac{1}{2\,y\,x - 1}\right] \&\mathrm{where}\ [y \neq 0] \tag{2.2.1.3}$$

Representing *inverse functions* in differential polynomial form is a starting point to tranform all these *inverse functions* into actual functions extending the expressiviness of the mathematical language.

\> *op*([1, 1], %), $y(0)$ = alpha

$$y' = -\frac{1}{2\,y\,x - 1},\ y(0) = \alpha \tag{2.2.1.4}$$

\> *dsolve*([%], $y(x)$, *series*)

$$y = \alpha + x + \alpha\,x^2 + \left(\frac{4\,\alpha^2}{3} + \frac{2}{3}\right)x^3 + \left(2\,\alpha^3 + \frac{5}{2}\,\alpha\right)x^4 + \left(\frac{104}{15}\,\alpha^2 + \frac{16}{15} + \frac{16}{5}\,\alpha^4\right)x^5 + \mathrm{O}\left(x^6\right) \tag{2.2.1.5}$$

\> $y(x) = (\mathrm{erf}@@(-1))(x)$

$$y = \mathrm{erf}^{(-1)}(x) \tag{2.2.1.6}$$

\> *dpolyform*(**(2.2.1.6)**, *no_Fn*)

$$\left[y'' = 2\,y\,y'^2\right] \&\mathrm{where}\ [y'' \neq 0] \tag{2.2.1.7}$$

\> *op*([1, 1], %), $y(0)$ = alpha, $\mathrm{D}(y)(0)$ = beta

$$y'' = 2\,y\,y'^2,\ y(0) = \alpha,\ \mathrm{D}(y)(0) = \beta \tag{2.2.1.8}$$

\> *dsolve*([%], $y(x)$, *series*)

$$y = \alpha + \beta\,x + \alpha\,\beta^2\,x^2 + \left(\frac{4}{3}\,\alpha^2\,\beta^3 + \frac{1}{3}\,\beta^3\right)x^3 + \left(\frac{7}{6}\,\beta^4\,\alpha + 2\,\alpha^3\,\beta^4\right)x^4 \tag{2.2.1.9}$$

$$+\left(\frac{46}{15}\beta^5\alpha^2+\frac{16}{5}\alpha^4\beta^5+\frac{7}{30}\beta^5\right)x^5+O(x^6)$$

>

The Differential Equations representing arbitrary algebraic expresssions

> *restart*; *with*(*PDEtools*) :

For ease of reading, these functions are declared to be displayed in a compact way. Also, derivatives are displayed as indexed objects.

> *declare*($f(x), g(x,y), \_F1(x,y), \_F2(x,y), \_F3(x,y), prime = x$)

$f(x)$ *will now be displayed as* f

$g(x,y)$ *will now be displayed as* g

$\_F1(x,y)$ *will now be displayed as* $\_F1$

$\_F2(x,y)$ *will now be displayed as* $\_F2$

$\_F3(x,y)$ *will now be displayed as* $\_F3$

derivatives with respect to x of functions of one variable will now be displayed with ' **(2.2.2.1)**

Consider the following non-polynomial expression.

> $f(x) = \tan(x)$

$$f = \tan(x) \tag{2.2.2.2}$$

> *dpolyform*(**(2.2.2.2)**, *no_Fn*)

$$[f' = f^2 + 1] \,\&\text{where}\, [f' \neq 0] \tag{2.2.2.3}$$

> $g(x,y) = \tan\left(2x - y^{\frac{1}{2}}\right)$

$$g = \tan\left(2x - \sqrt{y}\right) \tag{2.2.2.4}$$

A differential polynomial system (DPS) is given by:

> *dpolyform*(**(2.2.2.4)**, *no_Fn*)

$$\left[g_x = 2g^2 + 2,\ g_y^{\,2} = \frac{g^4}{4y} + \frac{g^2}{2y} + \frac{1}{4y}\right] \&\text{where} \left[g_y \neq 0,\ -g^2 - 1 \neq 0\right] \tag{2.2.2.5}$$

> *pdetest*(**(2.2.2.4)**, **(2.2.2.5)**)

$$[0, 0] \tag{2.2.2.6}$$

> EllipticF(exp(sqrt(z)), k)

$$F\left(e^{\sqrt{z}}, k\right) \tag{2.2.2.7}$$

> *FunctionAdvisor*(*DE*, **(2.2.2.7)**, [z, k])

$$\left[f(z,k) = F\left(e^{\sqrt{z}}, k\right), \left[f_{k,k,k} = \left(\frac{3 f_{k,z}}{f_z} + \frac{-5k^2+1}{k^3-k}\right) f_{k,k} + \left(\frac{(9k^2-3) f_k}{(k^3-k) f_z} + \frac{3 f(z,k)}{(k^2-1) f_z}\right) f_{k,z} + \frac{(-4k^2-1) f_k}{k^2 (k^2-1)}, f_{z,z} = \left(k^3 (k-1)^2 (k+1)^2 f_{k,k}^{\,2}\right.\right.\right. \tag{2.2.2.8}$$

$$+\left(\left(6k^6-8k^4+2k^2\right)f_k-2k(k-1)^2(k+1)^2 f_z+\left(2k^5-2k^3\right)f(z,k)\right)f_{k,k}+9k\left(k^2-\frac{1}{3}\right)^2 f_k^2+\left(\left(-6k^4+8k^2-2\right)f_z+\left(6k^4-2k^2\right)f(z,k)\right)f_k-4f_z^2kz+\left(-2k^3+2k\right)f(z,k)f_z+f(z,k)^2k^3\bigg)\bigg/\Big(4kz(k-1)^2(k+1)^2 f_{k,k}+\left(12k^4z-16k^2z+4z\right)f_k+\left(4k^3z-4kz\right)f(z,k)\Big), f_{k,z}^{\;2}=\frac{k^2 f_{k,k}^{\;2}}{4z}+\left(\frac{\left(3k^5-4k^3+k\right)f_k}{2z(k-1)^2(k+1)^2}+\frac{k^2\left(k^2-1\right)f(z,k)}{2z\left(k^4-2k^2+1\right)}\right)f_{k,k}+\frac{\left(-2k^3+2k\right)f_z f_{k,z}}{k^4-2k^2+1}+\frac{\left(3k^2-1\right)^2 f_k^2}{4z(k-1)^2(k+1)^2}+\frac{\left(3k^3-k\right)f(z,k)f_k}{2z(k-1)^2(k+1)^2}-\frac{f_z^2k^2}{k^4-2k^2+1}+\frac{k^2f(z,k)^2}{4z(k-1)^2(k+1)^2}\Bigg]$$

Not just known functions but also arbitrary functions of algebraic expressions can be represented in differential polynomial

> $F(x,y)=f(\exp(\mathrm{sqrt}(x))+g(y))$

$$F(x,y)=f\left(e^{\sqrt{x}}+g(y)\right) \qquad \textbf{(2.2.2.9)}$$

> $PDEtools\text{:-}dpolyform(\%, no\_Fn)$

$$\left[F_{x,y}=-\frac{F_x g_{y,y}}{g_y}+\frac{F_x F_{y,y}}{F_y}, F_{x,x}^{\;2}=\left(\frac{2F_x^{\;2}F_{y,y}}{F_y^{\;2}}-\frac{2F_x^{\;2}g_{y,y}}{F_y g_y}-\frac{F_x}{x}\right)F_{x,x}-\frac{F_x^{\;4}F_{y,y}^{\;2}}{F_y^{\;4}}+\left(\frac{2F_x^{\;4}g_{y,y}}{F_y^{\;3}g_y}+\frac{F_x^{\;3}}{F_y^{\;2}x}\right)F_{y,y}-\frac{F_x^{\;4}g_{y,y}^{\;2}}{F_y^{\;2}g_y^{\;2}}-\frac{F_x^{\;3}g_{y,y}}{F_y g_y x}+\frac{(x-1)F_x^{\;2}}{4x^2}\right]\&\text{where}\left[2F_{x,x}F_y^{\;2}g_y x-2F_{y,y}F_x^{\;2}xg_y+2g_{y,y}F_x^{\;2}xF_y+F_xg_yF_y^{\;2}\neq 0\right] \qquad \textbf{(2.2.2.10)}$$

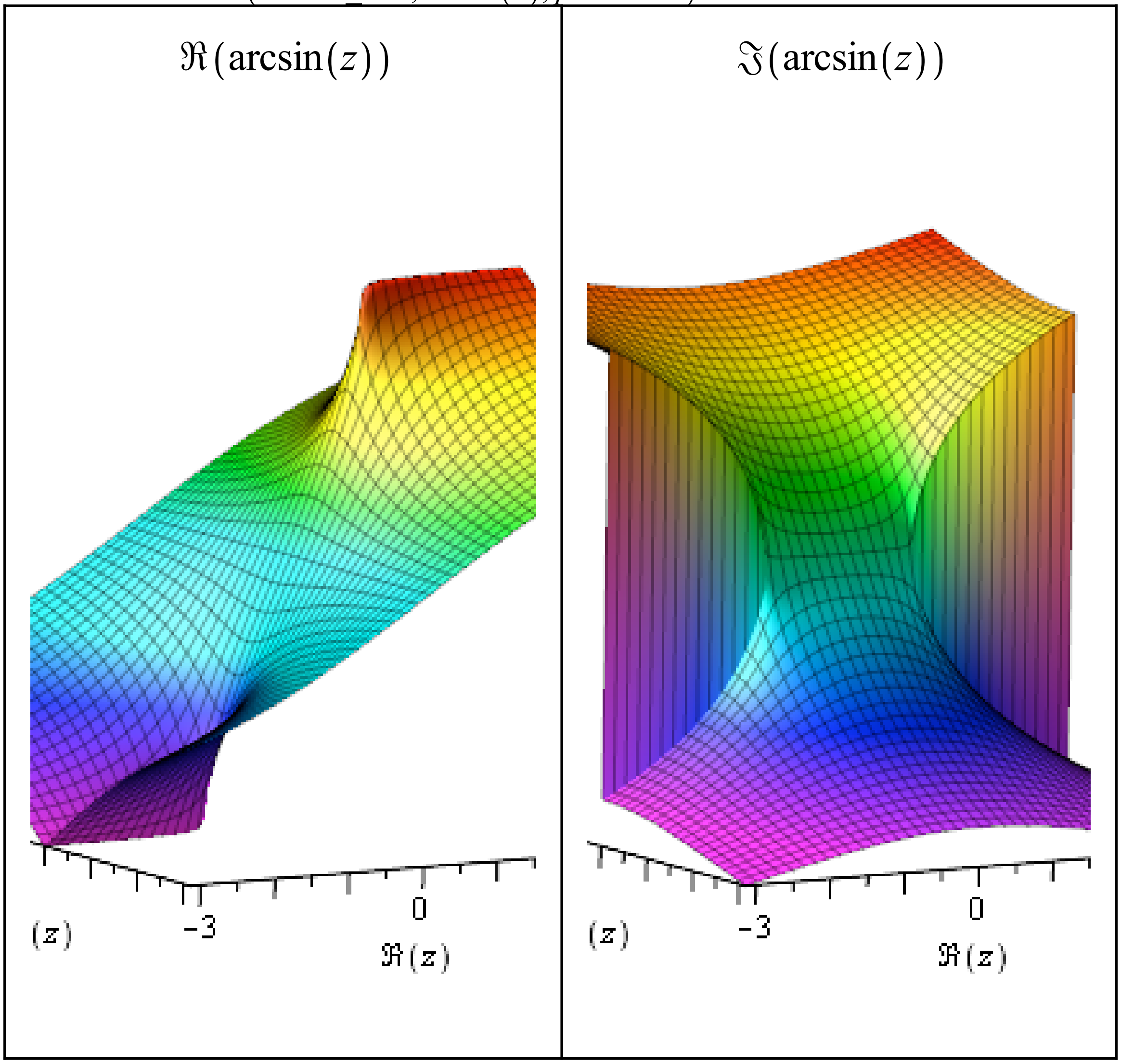
> $pdetest(\%\%, \%)$

$$[0,0] \qquad \textbf{(2.2.2.11)}$$

>

Branch cuts for algebraic expressions

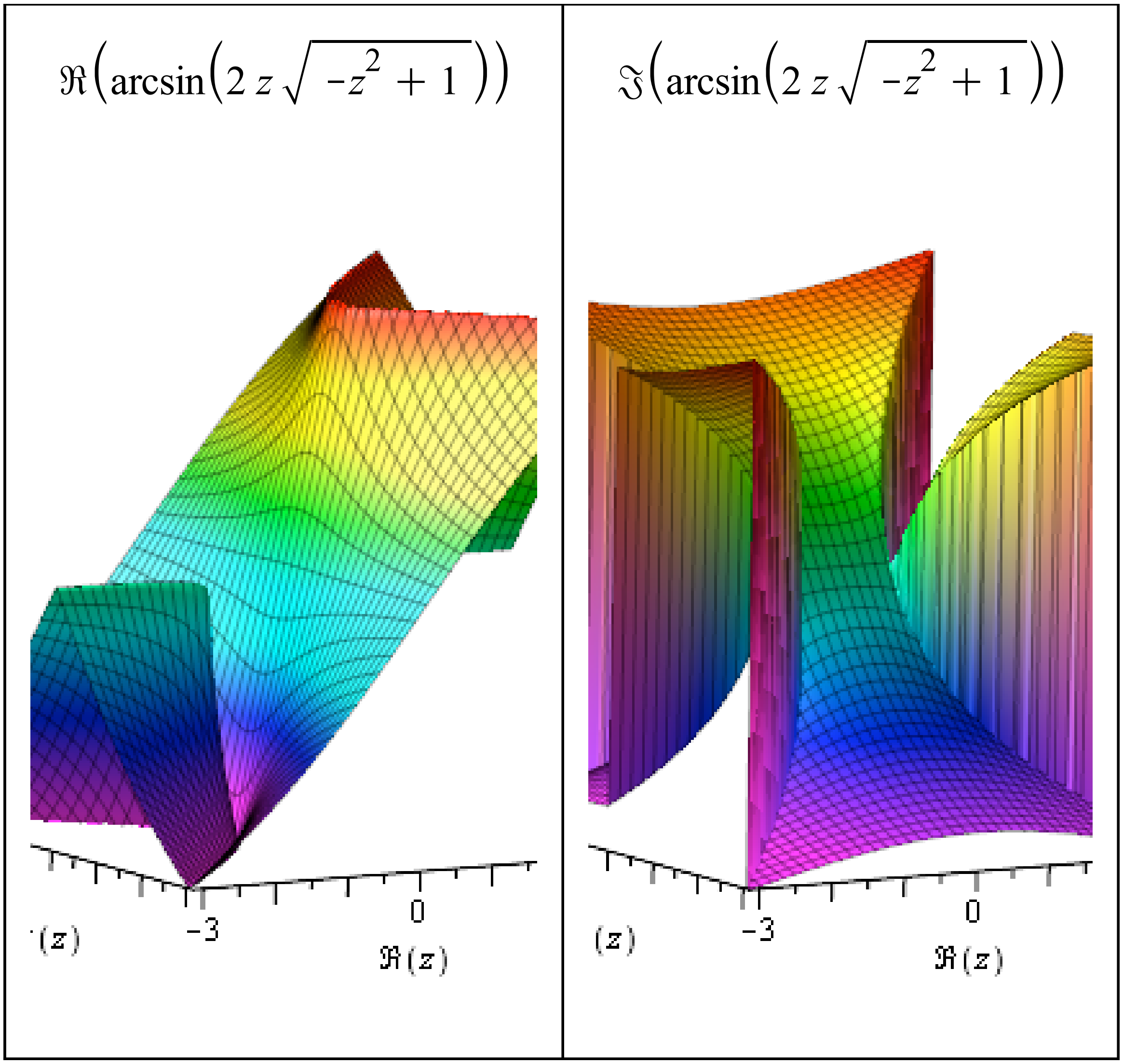
> *FunctionAdvisor*(*branch_cuts*, arcsin(z), *plot* = 3 D)



$[\arcsin(z), 1 \le z, z \le -1]$ **(2.3.1)**

What about the cuts of an algebraic expression?

> *FunctionAdvisor*$\left(branch\_cuts, \arcsin\left(2\,z\,\sqrt{1-z^2}\right), plot = 3\,\mathrm{D}\right)$

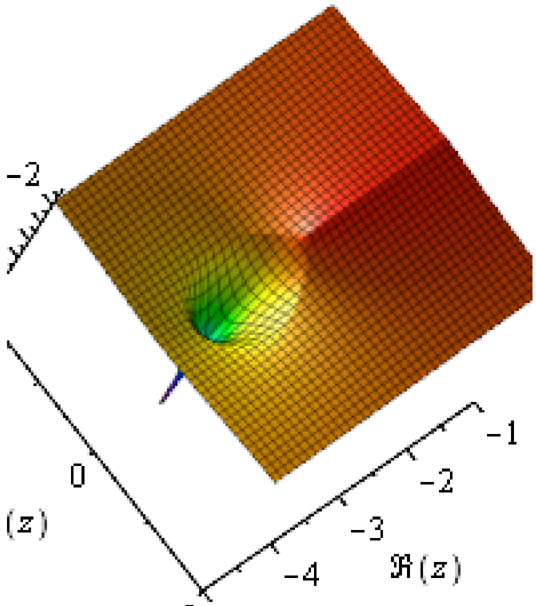
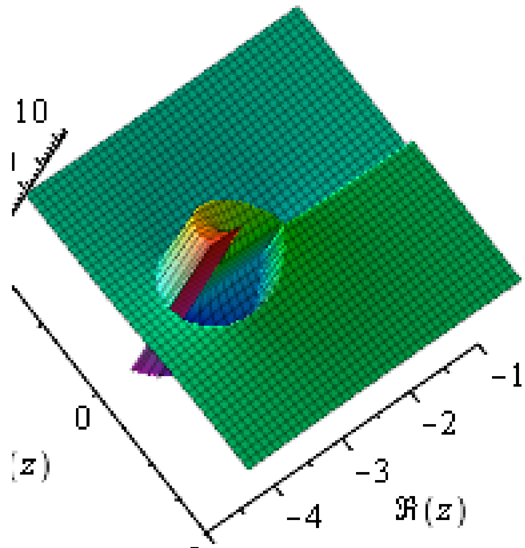


$$\left[\arcsin\left(2\,z\,\sqrt{-z^2+1}\right),\ 1 < z,\ z < -1,\ \Re(z) = -\frac{\sqrt{4\,\Im(z)^2+2}}{2},\ \Re(z) = \frac{\sqrt{4\,\Im(z)^2+2}}{2}\right] \tag{2.3.2}$$

A classic example in the theory of branch cut calculation is that of Kahan's teardrop:

$$> KTD := 2\,\mathrm{arccosh}\left(\frac{(3+(2\,z))}{3}\right) - 2\,\mathrm{arccosh}\left(\frac{(12+(5\,z))}{(3\,(z+4))}\right) - 2\,\mathrm{arccosh}\left(2\,(z+3)\,\sqrt{\frac{(z+3)}{(27\,(z+4))}}\right):$$

$$> \mathit{FunctionAdvisor}(\mathit{branch\_cuts}, \mathit{KTD}, \mathit{plot} = 3., \mathit{shift\_range} = -3, \mathit{scale\_range} = 2, \mathit{orientation} = [-124, 20, 14])$$

$\Re\left(2\,\mathrm{arccosh}\left(1+\frac{2}{3}\,z\right)\right.$ $-\,2\,\mathrm{arccosh}(1/$ $(3\,z+12)(12+5\,z))$ $-\,2\,\mathrm{arccosh}\Big(\,(2\,z$ 	$\Im\left(2\,\mathrm{arccosh}\left(1+\frac{2}{3}\,z\right)\right.$ $-\,2\,\mathrm{arccosh}(1/$ $(3\,z+12)(12+5\,z))$ $-\,2\,\mathrm{arccosh}\Big(\,(2\,z$ 

$$\Bigg[2\,\mathrm{arccosh}\left(1+\frac{2\,z}{3}\right)-2\,\mathrm{arccosh}\left(\frac{12+5\,z}{3\,z+12}\right)-2\,\mathrm{arccosh}\left((2\,z+6)\sqrt{\frac{z+3}{27\,z+108}}\right), z\le 0\wedge -4<z, -\frac{9}{2}<z\wedge z<-4, \Im(z)=\frac{\sqrt{-4\,\Re(z)^2-28\,\Re(z)-45}\;(\Re(z)+3)}{2\,\Re(z)+5}\wedge \Re(z)\le 0\wedge -\frac{9}{4}<\Re(z), \Im(z)=\frac{\sqrt{-4\,\Re(z)^2-28\,\Re(z)-45}\;(\Re(z)+3)}{2\,\Re(z)+5}\wedge \Re(z)\le -3\wedge -\frac{9}{2}<\Re(z), \Im(z)=-\frac{\sqrt{-4\,\Re(z)^2-28\,\Re(z)-45}\;(\Re(z)+3)}{2\,\Re(z)+5}\wedge \Re(z)\le 0\wedge -\frac{9}{4}<\Re(z), \Im(z)=\frac{\sqrt{-4\,\Re(z)^2-28\,\Re(z)-45}\;(\Re(z)+3)}{2\,\Re(z)+5}\wedge -\frac{5}{2}<\Re(z) \qquad \textbf{(2.3.3)}$$

$$\wedge \Re(z) < -\frac{9}{4}, \Im(z) = -\frac{\sqrt{-4\Re(z)^2 - 28\Re(z) - 45}\,(\Re(z)+3)}{2\Re(z)+5} \wedge \Re(z) \le -3 \wedge -\frac{9}{2} < \Re(z), \Im(z) = -\frac{\sqrt{-4\Re(z)^2 - 28\Re(z) - 45}\,(\Re(z)+3)}{2\Re(z)+5} \wedge -\frac{5}{2} < \Re(z) \wedge \Re(z) < -\frac{9}{4}\Big]$$

References

- Cheb-Terrab, E.S., England M., Bradford R., Davenport J.H., Wilson, D. "Branch cuts of algebraic expressions", ACM **Communications in Computer Algebra 48:1 pp. 24-27, ACM, 2014**.

- Cheb-Terrab, E.S. "The function wizard project: A Computer Algebra Handbook of Special Functions". **Proceedings of the Maple Summer Workshop**, University of Waterloo, Ontario, Canada, 2002.

>

The nth derivative problem for algebraic expressions

A power where the exponent is linear in the differentiation variable, a relatively easy problem, can be in a database

> *restart*

> $(\%diff = diff)\left(f^{\alpha z + \beta}, z\$n\right)$

$$\frac{\partial^n}{\partial z^n}\left(f^{\alpha z + \beta}\right) = \alpha^n f^{\alpha z + \beta} \ln(f)^n \quad \textbf{(2.4.1)}$$

More complicated, consider the k^{th} power of a generic function; the corresponding symbolic derivative can be mapped into a sum of symbolic derivatives of powers of $g(z)$ with lower degree

> $(\%diff = diff)\left(g(\alpha z + \beta)^k, z\$n\right)$ assuming $k :: posint, k > n$

$$\frac{\partial^n}{\partial z^n}\left(g(\alpha z + \beta)^k\right) = k \binom{-k+n}{n} \left(\sum_{\_k1=0}^{n} \frac{1}{k - \_k1}\left((-1)^{\_k1} \binom{n}{\_k1} g(\alpha z + \beta)^{k - \_k1} \frac{\partial^n}{\partial z^n}\left(g(\alpha z + \beta)^{\_k1}\right)\right)\right) \quad \textbf{(2.4.2)}$$

For $g = \ln$ this result can be expressed using Stirling numbers of the first kind

> $(\%diff = diff)\left(\ln(\alpha z + \beta)^k, z\$n\right)$

$$\frac{\partial^n}{\partial z^n}\left(\ln(\alpha z + \beta)^k\right) = \frac{\alpha^n \left(\sum_{\_k1=0}^{n} (k - \_k1 + 1)_{\_k1}\, S_n^{\_k1} \ln(\alpha z + \beta)^{k - \_k1}\right)}{(\alpha z + \beta)^n} \quad \textbf{(2.4.3)}$$

The case of the MeijerG function of an *arbitrary number of parameters*

> MeijerG([[a[i] $ i = 1 ..n], [b[i] $ i = n + 1 ..p]], [[b[i] $ i = 1 ..m], [b[i] $ i = m + 1 ..q]], z)

$$G^{m,\,n}_{p,\,q}\left(z\,\middle|\,\begin{matrix}a_1, \ldots, a_n, b_{n+1}, \ldots, b_p\\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q\end{matrix}\right) \tag{2.4.4}$$

> (%diff = diff)((2.4.4), z$k)

$$\frac{d^k}{dz^k} G^{m,\,n}_{p,\,q}\left(z\,\middle|\,\begin{matrix}a_1, \ldots, a_n, b_{n+1}, \ldots, b_p\\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q\end{matrix}\right) = G^{m,\,n+1}_{1+p,\,q+1}\left(z\,\middle|\,\begin{matrix}-k, a_1 - k, \ldots, a_n - k, b_{n+1} - k, \ldots, b_p - k\\ b_1 - k, \ldots, b_m - k, 0, b_{m+1} - k, \ldots, b_q - k\end{matrix}\right) \tag{2.4.5}$$

Not only the mathematics of this result is correct: the object returned is actually computable (for given concrete values of n, p, m and q)

Finally, the "holy grail" of this problem: the n^{th} derivative of a composite function $f(g(z))$ - an implementation of Faa di Bruno formula. Building on top of the basic blocks of knowledge, Faa di Bruno is now a new, higher level, basic block:

> (%diff = diff)(f(g(z)), z$n)

$$\frac{d^n}{dz^n} f(g(z)) = \sum_{\_k2=0}^{n} D^{(\_k2)}(f)(g(z))\,\mathrm{IncompleteBellB}\left(n, \_k2, \frac{d}{dz} g(z), \ldots, \frac{d^{n-\_k2+1}}{dz^{n-\_k2+1}} g(z)\right) \tag{2.4.6}$$

Note the symbolic sequence of symbolic order derivatives of lower degree, both of of f and g, also within the arguments of the IncompleteBellB function. This is a ***very*** abstract formula ...

To see this at work, consider, for instance, a case where the n^{th} derivatives of $f(z)$ and $g(z)$ can both be computed, say $f = \sin$, $g = \cos$

> sin(cos(α z + β))

$$\sin(\cos(\alpha z + \beta)) \tag{2.4.7}$$

> (%diff = diff)((2.4.7), z$n)

$$\frac{d^n}{dz^n} \sin(\cos(\alpha z + \beta)) = \sum_{\_k2=0}^{n} \sin\left(\cos(\alpha z + \beta) + \frac{\_k2\,\pi}{2}\right) \mathrm{IncompleteBellB}\left(n, \_k2, \cos\left(\alpha z + \beta + \frac{\pi}{2}\right)\alpha, \ldots, \cos\left(\alpha z + \beta + \frac{(n - \_k2 + 1)\,\pi}{2}\right)\alpha^{n-\_k2+1}\right) \tag{2.4.8}$$

Take for instance $n = 3$

> eval((2.4.8), n = 3)

$$\frac{d^3}{dz^3} \sin(\cos(\alpha z + \beta)) = \sum_{\_k2=0}^{3} \sin\left(\cos(\alpha z + \beta) + \frac{\_k2\,\pi}{2}\right) \mathrm{IncompleteBellB}\left(3, \_k2, \cos\left(\alpha z + \beta + \frac{\pi}{2}\right)\alpha, \ldots, \cos\left(\alpha z + \beta + \frac{(4 - \_k2)\,\pi}{2}\right)\alpha^{4-\_k2}\right) \tag{2.4.9}$$

> *value*((**2.4.9**))

$$\alpha^3 \sin(\alpha z+\beta)\cos(\cos(\alpha z+\beta)) - 3\alpha^3\cos(\alpha z+\beta)\sin(\alpha z+\beta)\sin(\cos(\alpha z+\beta)) + \alpha^3\sin(\alpha z+\beta)^3\cos(\cos(\alpha z+\beta)) = \alpha^3\sin(\alpha z+\beta)\cos(\cos(\alpha z+\beta)) - 3\alpha^3\cos(\alpha z+\beta)\sin(\alpha z+\beta)\sin(\cos(\alpha z+\beta)) + \alpha^3\sin(\alpha z+\beta)^3\cos(\cos(\alpha z+\beta)) \quad \textbf{(2.4.10)}$$

The two sides of this equation are equal

> *simplify*((*lhs* − *rhs*)((**2.4.10**)))

$$0 \quad \textbf{(2.4.11)}$$

>

Conversion network for mathematical and algebraic expressions

Examples

> *restart*;

Start with the error function

> erf(*z*)

$$\mathrm{erf}(z) \quad \textbf{(2.5.1.1)}$$

> *convert*((**2.5.1.1**), HermiteH)

$$-\frac{2\,H_{-1}(z)}{\sqrt{\pi}\,e^{z^2}} + 1 \quad \textbf{(2.5.1.2)}$$

> *convert*((**2.5.1.2**), KummerU)

$$-\frac{z\,U\left(1, \frac{3}{2}, z^2\right)\sqrt{z^2} - \sqrt{\pi}\,e^{z^2}\left(z - \sqrt{z^2}\right)}{\sqrt{z^2}\,\sqrt{\pi}\,e^{z^2}} + 1 \quad \textbf{(2.5.1.3)}$$

> *normal*(*convert*((**2.5.1.3**), WhittakerW))

$$-\frac{z\left(-\sqrt{\pi}\,e^{z^2}\left(z^2\right)^{1/4} + W_{-\frac{1}{4},\frac{1}{4}}\left(z^2\right)e^{\frac{z^2}{2}}\right)}{\sqrt{\pi}\,e^{z^2}\left(z^2\right)^{3/4}} \quad \textbf{(2.5.1.4)}$$

> *convert*((**2.5.1.4**), hypergeom)

$$-\frac{1}{\sqrt{\pi}\,e^{z^2}\left(z^2\right)^{3/4}}\left(z\left(-\sqrt{\pi}\,e^{z^2}\left(z^2\right)^{1/4} - 2\left(z^2\right)^{3/4}{}_1F_1\left(1; \frac{3}{2}; z^2\right) + \left(z^2\right)^{1/4}\sqrt{\pi}\,{}_0F_0(;; z^2)\right)\right) \quad \textbf{(2.5.1.5)}$$

> *simplify*((**2.5.1.5**))

$$\mathrm{erf}(z) \quad \textbf{(2.5.1.6)}$$

Many "rule conversions" can be requested at once and performed in a specified order; for instance, consider the following expression:

> $ee := \frac{1}{8}\, U\left(\frac{3}{2}, \frac{1}{2}, z^2\right) - \frac{1}{8}\, U\left(1, \frac{3}{2}, z^2\right) \sqrt{z^2} - \frac{1}{4}\, U\left(1, \frac{3}{2}, z^2\right) z^2 \sqrt{z^2} + \left(\frac{1}{4}\sqrt{z^2}\right)$

$$ee := \frac{U\left(\frac{3}{2}, \frac{1}{2}, z^2\right)}{8} - \frac{U\left(1, \frac{3}{2}, z^2\right)\sqrt{z^2}}{8} - \frac{U\left(1, \frac{3}{2}, z^2\right) z^2 \sqrt{z^2}}{4} + \frac{\sqrt{z^2}}{4} \qquad \textbf{(2.5.1.7)}$$

By converting it to 'KummerU' and applying rules for normalizing the first and second indices, here respectively called *a* and *b*, and then mixing them, you obtain varied representations for the same expression.

> *convert*(*ee*, KummerU, "normalize a", "normalize b", "mix a and b")

$$\frac{\sqrt{z^2}\, U\left(2, \frac{3}{2}, z^2\right)}{8} - \frac{U\left(\frac{1}{2}, \frac{1}{2}, z^2\right)}{8} - \frac{U\left(\frac{1}{2}, \frac{1}{2}, z^2\right) z^2}{4} + \frac{\sqrt{z^2}}{4} \qquad \textbf{(2.5.1.8)}$$

> *convert*((**2.5.1.8**), KummerU, "normalize a", "normalize b", "mix a and b")

$$\frac{(2 z^2 + 1)\, U\left(\frac{1}{2}, \frac{1}{2}, z^2\right)}{8} - \frac{U\left(1, \frac{3}{2}, z^2\right)\sqrt{z^2}}{8} - \frac{U\left(1, \frac{3}{2}, z^2\right) z^2 \sqrt{z^2}}{4} \qquad \textbf{(2.5.1.9)}$$

A further manipulation actually shows the expression was always equal to zero :)

> *convert*((**2.5.1.9**), KummerU, "normalize a", "normalize b", "mix a and b")

$$\frac{(2 z^2 + 1)\, U\left(\frac{1}{2}, \frac{1}{2}, z^2\right)}{8} - \frac{U\left(\frac{1}{2}, \frac{1}{2}, z^2\right)}{8} - \frac{U\left(\frac{1}{2}, \frac{1}{2}, z^2\right) z^2}{4} \qquad \textbf{(2.5.1.10)}$$

> *normal*((**2.5.1.10**))

$$0 \qquad \textbf{(2.5.1.11)}$$

>

General description

- The functions to which you can convert - the second argument could be one of these - are:

AiryAi	AiryBi	arccos	arccosh
arccot	arccoth	arccsc	arccsch
arcsec	arcsech	arcsin	arcsinh
arctan	arctanh	BellB	BesselI
BesselJ	BesselK	BesselY	ChebyshevT
ChebyshevU	cos	cosh	cot
coth	CoulombF	csc	csch
CylinderD	CylinderU	CylinderV	dilog

Ei	EllipticE	EllipticK	erf
exp	GAMMA	GegenbauerC	HankelH1
HankelH2	HermiteH	hypergeom	JacobiP
KummerM	KummerU	LaguerreL	LegendreP
LegendreQ	ln	MeijerG	polylog
Psi	sec	sech	sin
sinh	SphericalY	tan	tanh
WhittakerM	WhittakerW	Wrightomega	

- When a *function class* name is given as second argument the routines attempt to express expr using any of the functions of that class. The function classes understood by convert are:

`0F1`	`1F1`	`2F1`	Airy
arctrig	arctrigh	Bessel_related	Chebyshev
Cylinder	Ei_related	elementary	Elliptic_related
erf_related	GAMMA_related	Hankel	Heun
Kelvin	Kummer	Legendre	trig
trigh	Whittaker		

* (note spaces between words are filled with _)

- Despite the large number of function classes, most of the functions of mathematical physics belong to one of the three hypergeometric classes: 2F1, 1F1, and 0F1 where these three classes also include as particular cases all the elementary functions (trig, hyperbolic trig, their arcs, exp, and ln).

The Optional Arguments

The conversion routines accept an optional extra argument indicating a *rule*; it could be one rule or a sequence of them:

"raise a",	"lower a",	"normalize a",
"raise b",	"lower b",	"normalize b",
"raise c",	"lower c",	"normalize c",
"mix a and b",	"1F1 to 0F1",	"0F1 to 1F1"
"quadratic 1",	"quadratic 2",	"quadratic 3"
"quadratic 4",	"quadratic 5",	"quadratic 6"

"2a2b", "raise 1/2", "lower 1/2"

References

Cheb-Terrab, E.S. "The function wizard project: A Computer Algebra Handbook of Special Functions". **Proceedings of the Maple Summer Workshop**. University of Waterloo, Ontario, Canada, 2002.