Delay-differential equations

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1 Introduction to infinite-dimensional dynamical systems

All of the dynamical systems we have studied so far are finite-dimensional: The state at any time can be specified by listing a finite set of values. In today's lecture, we will consider infinite-dimensional systems. The most common types of infinite-dimensional dynamical systems involve the evolution of functions in time. For instance, if we want to study the evolution of chemical concentrations in time *and* space, we can phrase the problem as the change in time of the spatial distribution of the chemicals. This distribution is represented by a function of the spatial variables, i.e. $\mathbf{c} = \mathbf{c}(\mathbf{x})$. A function is, in a sense, an infinite set of values and so is an infinite-dimensional object.

Another way to think about the infinite dimensionality of function spaces is that their **basis set** is infinite. We can represent the coordinates of a point in an n-dimensional space as a superposition of n basis vectors, typically the unit vectors lying along the coordinate axes. On the other hand, we can't represent a general function, or even a function satisfying a few special conditions, using a finite superposition of some simple set of functions. Rather, the expansion of a general function requires an infinite basis set.

Example 1.1 Taylor's theorem implies that continuous, infinitely differentiable functions can be expanded in terms of the basis set $\{1, x, x^2, \ldots\}$. Only in exceptional cases (polynomials) does this expansion terminate.

Example 1.2 The set of functions which are zero at $x = 0$ and at $x = 1$ can be represented as a superposition of the functions $\{s_1, s_2, s_3, \ldots\}$ where $s_n = \sin(n\pi x)$. Most functions are of course not representable exactly as a finite superposition of sine waves.

As you can imagine, the analysis of infinite-dimensional dynamical systems poses some difficulties. Nevertheless, we can adapt techniques developed in the finite-dimensional case to at least some of the common types infinite-dimensional systems. In particular, linear stability analysis continues to be a useful technique, although we have to work a little harder at it.

2 Introduction to delay-differential equations

Delay-differential equations (DDEs) are a large and important class of dynamical systems. They often arise in either natural or technological control problems. In these systems, a controller monitors the state of the system, and makes adjustments to the system based on its observations. Since these adjustments can never be made instantaneously, a delay arises between the observation and the control action.

There are different kinds of delay-differential equations. We will focus on just one kind, namely those of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\tau_1), \mathbf{x}(t-\tau_2), \ldots, \mathbf{x}(t-\tau_n)), \tag{1}$$

where the quantities τ_i are positive constants. In other words, we will focus on equations with fixed, discrete delays. There are other possibilities, notably equations with state-dependent delays (the τ_i's depend on $\mathbf{x}$) or with distributed delays (the right-hand side of the differential equation is a weighted integral over past states).

When we give initial conditions for finite-dimensional dynamical systems, we only need to specify a small set of numbers, namely the initial values of the state variables, and perhaps the initial time in nonautonomous systems. In order to solve a delay equation, we need more: At every time step, we have to look back to earlier values of $\mathbf{x}$. We therefore need to specify an **initial function** which gives the behavior of the system prior to time 0 (assuming that we start at $t = 0$). This function has to cover a period at least as long as the longest delay since we will be looking back in time that far.

Let us for the moment specialize further to equations with a single delay, i.e.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\tau)). \tag{2}$$

The initial function would be a function $\mathbf{x}(t)$ defined on the interval $[-\tau, 0]$. How are we to understand the dynamics induced by this delay equation? We could think in the same terms as we do for ordinary differential equations, namely that the solution consists of a sequence of values of $\mathbf{x}$ at increasing values of t. From a purely theoretical perspective however, this is not the best way to think of equations of this type. A much better way is to think of the solution of this DDE as a mapping from functions on the interval $[t-\tau, t]$ into functions on the interval $[t, t+\tau]$. In other words, the solutions of this dynamical system can be thought of as a sequence of functions $\mathbf{f}_0(t), \mathbf{f}_1(t), \mathbf{f}_2(t), \ldots$, defined over a set of contiguous time intervals of length τ. The points $t = 0, \tau, 2\tau, \ldots$ where the solution segments meet are called **knots**. Figure 1 illustrates this mapping. Existence and uniqueness theorems analogous to those for ordinary differential equations are much more easily proven in this conceptual framework than by trying to think of DDEs as an evolution over the state space $\mathbf{x}$.

In some very simple cases, we can work out this mapping analytically, as the following example demonstrates.

Example 2.1 Consider the delay-differential equation

$$\dot{x} = -x(t-1). \tag{3}$$

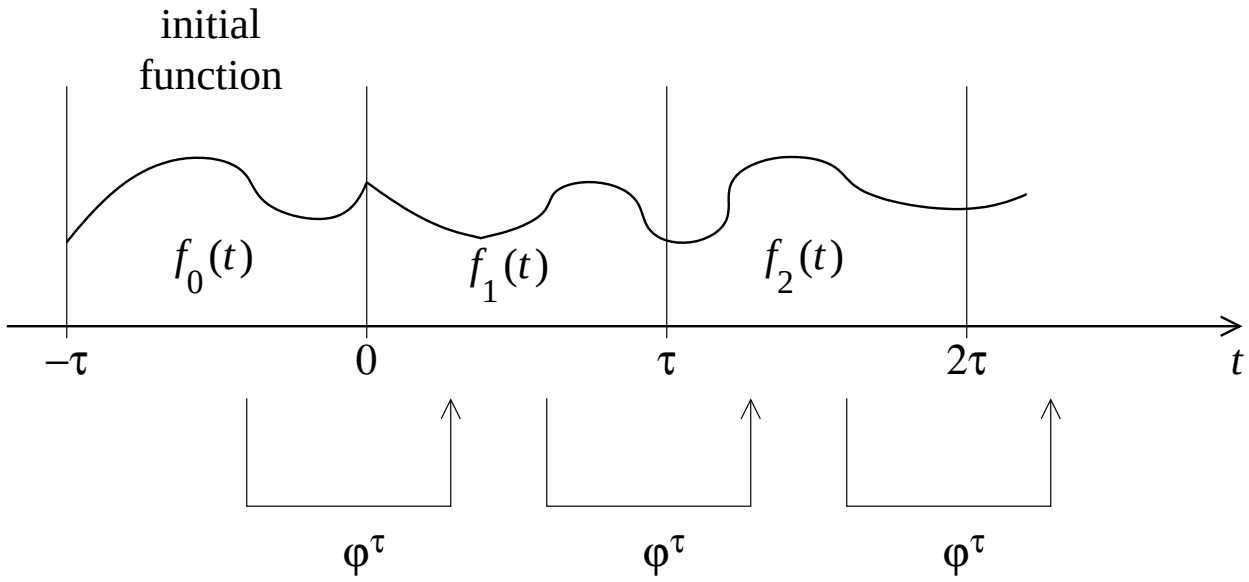

Figure 1: The action of the evolution operator φ^τ for a delay-differential equation of the form 2 is to take a function defined over a time interval of length τ and to map it into another function defined over a similar time interval. For instance, the initial function defined over the interval $[-\tau, 0]$ is mapped into a solution curve on the interval $[0, \tau]$.

Suppose that we have $x(t) = f_{i-1}(t)$ over some interval $[t_i - 1, t_i]$. Then, over the interval $[t_i, t_i + 1]$, we have, by separation of variables,

$$\begin{aligned} \int_{f_{i-1}(t_i)}^{x(t)} dx' &= -\int_{t_i}^{t} f_{i-1}(t'-1)\, dt'. \\ \therefore x(t) &= f_i(t) = f_{i-1}(t_i) - \int_{t_i}^{t} f_{i-1}(t'-1)\, dt'. \end{aligned}$$

The foregoing example provides the basic theory for the simplest method for solving DDEs, known as the **method of steps**. The following example demonstrates its use to obtain a solution given some initial data.

Example 2.2 Suppose that we are given the problem of solving equation 3 given the constant initial data

$$x(t) = 1 \qquad \text{for} \qquad t \in [-1, 0].$$

In the interval [0,1], we have

$$x(t) = 1 - \int_0^t 1\, dt' = 1 - t.$$

In the interval [1,2], we have

$$x(t) = 0 - \int_1^t \left[1 - (t'-1)\right] dt' = -\left[2t - \frac{1}{2}t^2\right]_1^t = -2t + \frac{1}{2}t^2 + \frac{3}{2}.$$

On [2,3], the solution is

$$\begin{aligned} x(t) &= -\frac{1}{2} - \int_2^t \left[-2(t'-1) + \frac{1}{2}(t'-1)^2 + \frac{3}{2} \right] dt' \\ &= -\frac{1}{2} - \left[-(t'-1)^2 + \frac{1}{6}(t'-1)^3 + \frac{3}{2}t' \right]_2^t \\ &= \frac{5}{3} + (t-1)^2 - \frac{1}{6}(t-1)^3 - \frac{3}{2}t. \end{aligned}$$

We can of course automate these calculations fairly easily in Maple. The following Maple procedure computes the f_i's defined in the previous example, f_0 being the initial function:

```
> f := proc(i,t) option remember;
> if i=0 then 1
> else f(i-1,i-1) - int(f(i-1,xi-1),xi=i-1..t);
> fi; end;
```

This routine depends on the fact that the delay is 1 so that the ends of the intervals over which each of the f_i's are valid are just $t = 0, 1, 2, \ldots$ To plot the result, we use the following Maple commands:

```
> with(plots):
> for i from 0 to 20 do
> p[i] := plot(f(i,t),t=i-1..i): od:
> display([p[j] $ j=0..20]);
```

The result is shown in Fig. 2. The solution is a damped oscillation made up in piecewise fashion by a set of polynomials, the functions f_i.

If you look at the solution of the simple DDE plotted in Fig. 2, you may notice that the first derivative of $x(t)$ isn't continuous at the first knot, $t = 0$. This isn't surprising: For $t < 0$, we have an arbitrary initial function. For $t > 0$, the solution is dictated by the differential equation. Since we made no special attempt to match up the initial function to the solution, the derivative given by equation 3 was bound not to match up with the zero derivative of our initial function. It is of course possible to cook up an initial function whose first derivative at $t = 0$ matches that of the solution ($f_0(t) = at$ works in this case, for any value of a), but then the second derivative won't. If you somehow manage to match up the second derivative, some higher derivative will be discontinuous at $t = 0$ except in some highly artificial circumstances in which you can actually find a particular solution to the DDE. In these cases, you can use this solution as the initial function, and of course all will be smooth and continuous. In general however, the derivatives of $x(t)$ at $t = 0$ will be discontinuous. These discontinuities propagate: If the first derivative is discontinuous at $t = 0$, then the second derivative will be discontinuous at $t = \tau$ since $\ddot{x}(t)$ is related by the DDE to $\dot{x}(t-\tau)$. For instance, for equation 3, $\ddot{x}(t) = -\dot{x}(t-\tau)$ so a discontinuity in the first derivative at $t = 0$ becomes a discontinuity in the second derivative at $t = \tau$, then a discontinuity in the third derivative at $t = 2\tau$, and so on.

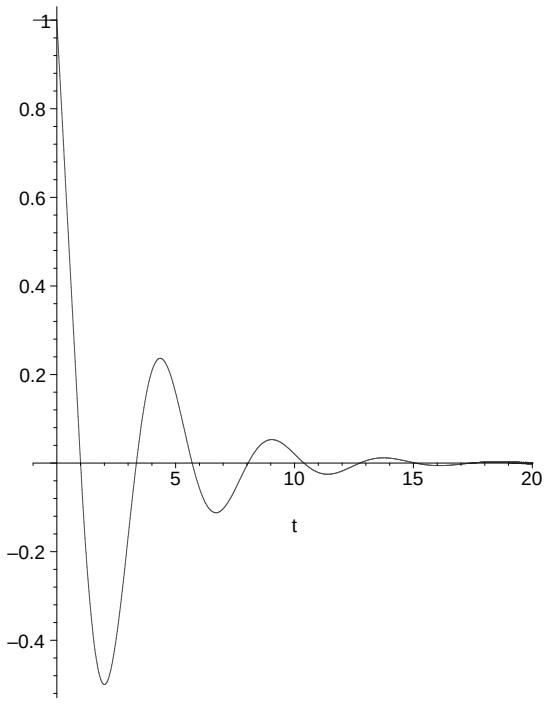

Figure 2: Solution of the DDE $\dot{x} = -x(t-\tau)$ obtained in Maple by the method of steps.

Most DDEs don't have analytic solutions, so it is generally necessary to resort to numerical methods. Because the solutions have discontinuous derivatives at the knots, it is necessary to be a little careful when using numerical methods designed for ODEs with DDEs. ODE integrators generally assume that at least the first few derivatives are continuous. They can therefore go badly wrong at the knots. On the other hand, I have generally found that you can get meaningful answers, even from integrators which shouldn't in theory work, provided you're careful. You should *always* check results of numerical integrations of DDEs by reducing the size of the time step and checking at least a few of your results with a second integration method. If the results are in reasonable agreement, then you're probably in good shape. Come to think of it, this isn't bad advice for ODEs, either.

3 Linearized stability analysis

As with other types of equations, we derive a lot of insight from a stability analysis of the equilibria. An equilibrium point is a point in the state space for which $\mathbf{x}(t) = \mathbf{x}^*$ is a solution for all t. Thus, for a DDE of the form 1, equilibrium points satisfy

$$\mathbf{f}(\mathbf{x}^*, \mathbf{x}^*, \mathbf{x}^*, \ldots, \mathbf{x}^*) = 0.$$

When we work out the stability of the equilibrium points of ODEs, we assume that the system has been displaced by a small distance in phase space from the equilibrium. With ODEs, the phase space is a finite-dimensional coordinate space, so both conceptually and operationally, this is pretty

simple. We proceed in an analogous way with DDEs, except that the phase space is now our infinite-dimensional function space, so that we have to consider displacements from equilibrium in this space. In other words, our displacements are time-dependent functions $\delta\mathbf{x}(t)$ persisting over an interval of at least $\tau_{\max}$, the longest delay.[1]

Before we proceed, it is convenient to introduce some new notation. Writing things like $\mathbf{x}(t-\tau)$ quickly becomes tedious. It is a common convention among people who work with DDEs that variables without subscripts are instantaneous (i.e. $\mathbf{x} \equiv \mathbf{x}(t)$) while delayed variables are indicated by the subscripted value of the delay ($\mathbf{x}_\tau \equiv \mathbf{x}(t-\tau)$). We use this convention below.

Let $\mathbf{x}^*$ be an equilibrium of equation 1, and let the system be disturbed from equilibrium by a small perturbation which lasts from $t = t_0 - \tau_{\max}$ to t_0. Let $\delta\mathbf{x}(t)$ be the displacement from equilibrium, assumed small, at any time in the open interval $[t_0 - \tau_{\max}, \infty)$. Accordingly,

$$\mathbf{x} = \mathbf{x}^* + \delta\mathbf{x},$$

and

$$\dot{\mathbf{x}} = \dot{\delta\mathbf{x}} = \mathbf{f}(\mathbf{x}^* + \delta\mathbf{x}, \mathbf{x}^* + \delta\mathbf{x}_{\tau_1}, \mathbf{x}^* + \delta\mathbf{x}_{\tau_2}, \ldots, \mathbf{x}^* + \delta\mathbf{x}_{\tau_n}).$$

Since each of the quantities $\delta\mathbf{x}, \delta\mathbf{x}_{\tau_1}, \delta\mathbf{x}_{\tau_2}, \ldots, \delta\mathbf{x}_{\tau_n}$ is small, we can linearize the differential equation about the equilibrium point using our usual Taylor series method:

$$\dot{\delta\mathbf{x}} \approx \mathbf{J}_0\delta\mathbf{x} + \mathbf{J}_{\tau_1}\delta\mathbf{x}_{\tau_1} + \mathbf{J}_{\tau_2}\delta\mathbf{x}_{\tau_2} + \ldots + \mathbf{J}_{\tau_n}\delta\mathbf{x}_{\tau_n}. \tag{4}$$

To obtain this equation, we used the fact that $\mathbf{f}(\mathbf{x}^*, \mathbf{x}^*, \mathbf{x}^*, \ldots, \mathbf{x}^*) = 0$. The quantity $\mathbf{J}_0$ is the usual Jacobian with respect to $\mathbf{x}$ evaluated at the equilibrium point, while the matrices $\mathbf{J}_{\tau_i}$ are the Jacobians with respect to the corresponding $\mathbf{x}_{\tau_i}$ again evaluated at $\mathbf{x} = \mathbf{x}_{\tau_1} = \mathbf{x}_{\tau_2} = \ldots = \mathbf{x}_{\tau_n} = \mathbf{x}^*$.

In linear ODEs, the solutions are exponential functions of time, with exponents given by the eigenvalues of the Jacobian matrix. Suppose that the linear DDE 4 also has exponential solutions, i.e. that we can write

$$\delta\mathbf{x}(t) = \mathbf{A}e^{\lambda t}.$$

Substituting this ansatz into equation 4 and rearranging a bit, we get

$$\lambda\mathbf{A} = \left(\mathbf{J}_0 + e^{-\lambda\tau_1}\mathbf{J}_{\tau_1} + e^{-\lambda\tau_2}\mathbf{J}_{\tau_2} + \ldots + e^{-\lambda\tau_n}\mathbf{J}_{\tau_n}\right)\mathbf{A}.$$

Linear algebraic theory tells us that this equation can only be satisfied with nonzero displacement amplitudes $\mathbf{A}$ if

$$\left|\mathbf{J}_0 + e^{-\lambda\tau_1}\mathbf{J}_{\tau_1} + e^{-\lambda\tau_2}\mathbf{J}_{\tau_2} + \ldots + e^{-\lambda\tau_n}\mathbf{J}_{\tau_n} - \lambda\mathbf{I}\right| = 0, \tag{5}$$

where $\mathbf{I}$ is the identity matrix. Equation 5 is called the **characteristic equation** of the equilibrium point.

Equation 5 looks a little like an ordinary eigenvalue problem, except for the appearance of the exponential terms. If we expand out the determinant, we will get equations that have polynomial parts, but which also include some terms in $e^{\lambda\tau_i}$. These are called **quasi-polynomials**. In principle, our job is the same: If any of the solutions of the characteristic equation have positive real parts,

[1] Actual perturbations may be shorter, but the equilibrium isn't stable unless it can survive small displacements along the entire immediately relevant portion of its history, i.e. going back to $t - \tau_{\max}$.

then the equilibrium point is unstable. If they all have negative real parts, the equilibrium point is stable. If the leading characteristic values are zero, then the stability is undecidable to linear order. The polynomial equations we have to solve in the ODE stability problem are nice: A polynomial of degree *n* has exactly *n* complex roots. We can therefore get all the roots, at least in principle, and look at each one to determine the stability of an equilibrium. On the other hand, quasi-polynomials usually have an *infinite number* of roots in the complex plane. We can't find all the roots, so we'll have to use different strategies to work out the stability of equilibrium points. Unfortunately, there isn't one universal strategy, so we'll have to just work through some examples.

Example 3.1 Let us start by analyzing the stability of equation 3. Since it's a single equation, the matrices become scalars, so things should be fairly simple. The delay is $\tau = 1$. The steady state is clearly $x^* = 0$. The "Jacobians" are

$$\begin{aligned} J_0 &= 0, \\ \text{and} \qquad J_1 &= -1. \end{aligned}$$

Note that the bars in equation 5 symbolize the determinant operation, and that the determinant of a single number is just that number itself. The characteristic equation is therefore[2]

$$\chi(\lambda) = e^{-\lambda} + \lambda = 0.$$

$\chi(\lambda)$ has an absolute minimum of 1 at $\lambda = 0$, so the characteristic equation has no real solutions. To find complex solutions, we write

$$\lambda = \mu + i\nu,$$

where μ and ν are, respectively, the real and imaginary parts of λ. Substituting into the characteristic equation, we get

$$\begin{aligned} e^{-\mu}e^{-i\nu} + \mu + i\nu &= 0. \\ \therefore e^{-\mu}(\cos\nu - i\sin\nu) + \mu + i\nu &= 0. \\ \therefore e^{-\mu}\cos\nu + \mu + i\left(\nu - e^{-\mu}\sin\nu\right) &= 0. \end{aligned}$$

If a complex value is equal to zero, then both the real and imaginary parts have to be zero. Thus in this case,

$$e^{-\mu}\cos\nu = -\mu, \tag{6a}$$

$$\text{and} \qquad e^{-\mu}\sin\nu = \nu. \tag{6b}$$

We want to find out whether these equations can have solutions with positive values of the real part μ. Note that the characteristic values come in complex-conjugate pairs. In other words, if (μ, ν) is a solution of equations 6, then so is $(\mu, -\nu)$, as you can verify by substituting the latter into the equations. We can therefore restrict our attention to positive values of ν.

[2] I changed the signs of both terms to avoid carrying unnecessary negative signs around through what follows.

Now suppose that there are solutions with positive μ. According to equation 6a, we must therefore have $\cos\nu < 0$. This means that $\nu > \pi/2$, since for any smaller, positive value of ν, $\cos\nu$ is positive. On the other hand, $e^{-\mu} < 1$ if $\mu > 0$, and $|\sin\nu| < 1$, so that equation 6b implies that $|\nu| < 1$. This is a contradiction, since ν cannot simultaneously be larger than $\pi/2$ and smaller in magnitude than 1. The real part of the characteristic value therefore *cannot* be positive, so the equilibrium point is stable.

Example 3.2 Our rate of breathing is mostly controlled by arterial carbon dioxide levels. Michael Mackey and Leon Glass of McGill University developed a heuristic model of this control system back in the late 1970's which has proven to be quite useful [1]. They assumed that metabolic activity produces carbon dioxide at a constant rate λ. Removal of carbon dioxide from the bloodstream is proportional both to the current concentration of carbon dioxide, x, and to the ventilation (volume of gas exchanged by the lungs per unit time). The latter is the quantity controlled by the level of carbon dioxide in the blood. The control system is complex and involves detection of carbon dioxide levels by receptors in the brainstem. The response in the ventilation V is sigmoidal and can be modeled by the equation

$$V = \frac{V_{\max} x^n}{\theta^n + x^n},$$

where $V_{\max}$, θ and n are constants. This detection, and the subsequent adjustment to the ventilation, take time. Accordingly, the ventilation is really a function of the delayed concentration of carbon dioxide. The net rate of change of the concentration of carbon dioxide is therefore

$$\frac{dx}{dt} = \lambda - \alpha x V_\tau,$$

where α is a proportionality constant. Explicitly, our differential equation is

$$\frac{dx}{dt} = \lambda - \alpha x \frac{V_{\max} x_\tau^n}{\theta^n + x_\tau^n}.$$

With the change of variables $y = x/\theta$, $z = \lambda t/\theta$, $a = \alpha\theta V_{\max}/\lambda$ and $\xi = \lambda\tau/\theta$, this equation can be put in the dimensionless form

$$\dot{y} = 1 - \frac{a y y_\xi^n}{1 + y_\xi^n},$$

where the dot represents the derivative with respect to the dimensionless time z.

The steady state satisfies

$$\frac{(y^*)^{n+1}}{1 + (y^*)^n} = \frac{1}{a}. \tag{7}$$

Note that the function on the left-hand side is zero when $y^* = 0$ and that

$$\frac{d}{dy^*}\left(\frac{(y^*)^{n+1}}{1 + (y^*)^n}\right) = \frac{(y^*)^n (n + 1 + (y^*)^n)}{(1 + (y^*)^n)^2} > 0.$$

Since the derivative is strictly positive and tends to a constant at large n, this function passes through every positive value and so there is a unique, positive steady state for each value of n and a. We can't determine the value of this steady state without knowing these two parameters. Hopefully that won't be too much of a problem.

The (one-dimensional) Jacobians are

$$J_0 = -\left.\frac{a y_\xi^n}{1+y_\xi^n}\right|_{y_\xi=y^*} = \frac{-a(y^*)^n}{1+(y^*)^n}. \tag{8a}$$

$$J_\xi = -ay\left.\frac{n y_\xi^{n-1}}{\left(1+y_\xi^n\right)^2}\right|_{y=y^*,y_\xi=y^*} = \frac{-an(y^*)^n}{[1+(y^*)^n]^2}. \tag{8b}$$

Note that both J_0 and J_ξ are negative constants.

The characteristic equation is

$$J_0 + e^{-\lambda\xi}J_\xi - \lambda = 0.$$

We will again write $\lambda = \mu + i\nu$ and separate the real and imaginary parts of this equation. We get

$$J_0 + e^{-\mu\xi}J_\xi\cos(\nu\xi) - \mu = 0, \tag{9a}$$

$$\text{and} \qquad e^{-\mu\xi}J_\xi\sin(\nu\xi) + \nu = 0. \tag{9b}$$

In this case, no simple argument rules out positive values of μ. On the other hand, negative values of μ are clearly possible: Suppose that the delay ξ is very small. Then, $\sin(\nu\xi) \to 0$, so that equation 9b implies $\nu \approx 0$. Equation 9a becomes $\mu \approx J_0 + J_\xi$. Since the two Jacobians are negative, μ is then negative.

If we know that the real part of the characteristic value can be negative, the standard trick to deal with these problems is to go looking for the bifurcation directly. In other words, set $\mu = 0$ and see if we can eliminate ν to get a sensible bifurcation condition. Setting $\mu = 0$, equations 9 become

$$J_\xi\cos(\nu\xi) = -J_0, \tag{10a}$$

$$\text{and} \qquad J_\xi\sin(\nu\xi) = -\nu. \tag{10b}$$

We can't satisfy the first of these equations if $\nu = 0$ since J_0 and J_ξ have the same sign. Accordingly, a pair of complex eigenvalues will cross the imaginary axis together, so we are looking for an Andronov-Hopf bifurcation.

Continuing, if we square these two equations and add them, using the fact that $\sin^2 u + \cos^2 u = 1$, we get

$$J_\xi^2 = J_0^2 + \nu^2. \tag{11}$$

Since $\nu^2 \geq 0$, this equation has real solutions if and only if

$$|J_\xi| \geq |J_0|.$$

Using equations 8, this inequality becomes

$$n \geq 1 + (y^*)^n. \tag{12}$$

We see immediately that $n > 1$ is a necessary, but certainly not sufficient, condition for the bifurcation to occur. The question we now face is whether this inequality can be satisfied, and if so whether we need to impose additional conditions.

Inequality 12 can be satisfied provided y^* isn't too large. To see this, imagine that y^* is very small. Then the right-hand side of the inequality is actually a decreasing function of n, and n can eventually be made sufficiently large to satisfy the inequality. If on the other hand y^* is very large, $1 + (y^*)^n$ will increase too fast for n to catch up. The boundary case is obtained when the line $f(n) = n$ is just tangent to $g(n) = 1 + (y^*)^n$, or in other words when both the values of these functions and their tangents are equal:

$$\begin{aligned} n &= 1 + (y^*)^n, \\ \text{and} \qquad 1 &= y^n \ln y. \end{aligned}$$

If we solve these two equations for n and y^* in Maple, we find $n = 4.5911$ and $y^* = 1.3211$. This means that we must have $y^* < 1.3211$ in order for the bifurcation to be possible.

We can do a little better still. Equation 10a can be solved for ν:

$$\nu = \frac{1}{\xi} \arccos(-J_0/J_\xi).$$

If we now substitute this equation into 11 and rearrange, we get

$$\xi = \frac{\arccos(-J_0/J_\xi)}{\sqrt{J_\xi^2 - J_0^2}}. \tag{13}$$

We know that the system is stable if ξ is small, so we should get an Andronov-Hopf bifurcation at this value of ξ, with limit cycle oscillations for ξ above this boundary.

Equation 13 implicitly defines the bifurcation curves in (e.g.) the (n, ξ) parameter plane. The overall strategy for drawing these curves can be described as follows:

1. Choose a value of a.
2. For each n in the range of interest,
   (a) Calculate y^* and verify if inequality 12 is satisfied. If so, proceed. Otherwise, try a larger value of n.
   (b) Calculate J_0 and J_ξ.
   (c) Calculate the bifurcation value of ξ from equation 13.

The Maple code which carries out this procedure follows:

```
> a := 5;
> ystar := n -> fsolve(y^(n+1)/(1+y^n)=1/a,y);
> J0 := n -> -a*ystar(n)^n/(1+ystar(n)^n);
```

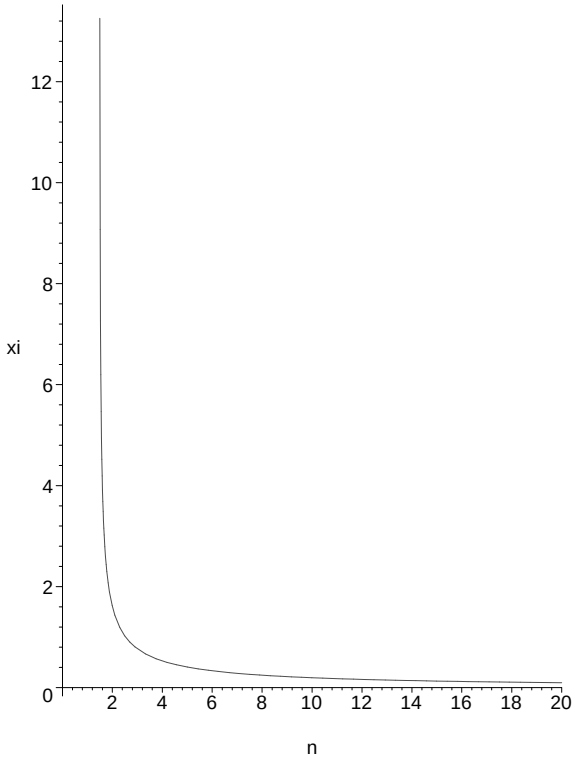

Figure 3: Plot of the Andronov-Hopf bifurcation curve for the Mackey-Glass model with $a = 5$. The system is oscillatory for sufficiently large values of the delay, i.e. for ξ above this curve.

```
> Jxi := n -> -a*n*ystar(n)^n/(1+ystar(n)^n)^2;
> xi := n -> if (n>1+ystar(n)^n) then
> arccos(-J0(n)/Jxi(n))/sqrt(Jxi(n)^2-J0(n)^2) else undefined fi;
> plot(xi,0..20,labels=['n','xi']);
```

The value $a = 5$, as well as the plotting range for n were chosen more or less on a whim. The graphical output is shown in Fig. 3.

Finally note that we can confirm the results of this analysis by simulation in `xpp`. The following is a simple `xpp` input file to simulate the Mackey-Glass model:

```
# Mackey-Glass model
y' = 1 - a*y*delay(y,xi)^n/(1+delay(y,xi)^n)

param a=5, xi=0.6, n=4

@ DELAY=50, MAXSTOR=1000000

done
```

In `xpp`, $x(t-\xi)$ is entered as `delay(x,xi)`. Note also the setting of the internal parameter `DELAY`: `xpp` needs to store the history of the solution in order to look up delayed values. The value of `DELAY` tells it for how long to hold the past values. The delay-differential equation

integrator *will not work* if the delay, in our case `xi`, is equal to or larger than the value of `DELAY`.

The examples covered in this section were both of systems with a single variable. Adding variables makes the analysis somewhat harder, but adding delays really makes things difficult. Also note that the automated bifurcation analysis of delay-differential equations is still in its infancy and is not available in AUTO.

References

[1] M. C. Mackey and L. Glass, Science **197**, 287 (1977).