13.2.2 Variational Approach

To derive the equation of motion of a membrane using the extended Hamilton's principle, the expressions for the strain and kinetic energies as well as the work done by external forces are needed. The strain and kinetic energies of a membrane can be expressed as

$$\pi = \frac{1}{2} \iint_A P \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] dA \tag{13.8}$$

$$T = \frac{1}{2} \iint_A \rho \left(\frac{\partial w}{\partial t} \right)^2 dA \tag{13.9}$$

The work done by the distributed pressure loading $f(x, y, t)$ is given by

$$W = \iint_A f w \, dA \tag{13.10}$$

The application of Hamilton's principle gives

$$\delta \int_{t_1}^{t_2} (\pi - T - W) \, dt = 0 \tag{13.11}$$

or

$$\delta \int_{t_1}^{t_2} \left\{ \frac{1}{2} \iint_A P \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] dA - \frac{1}{2} \iint_A \rho \left(\frac{\partial w}{\partial t} \right)^2 dA - \iint_A f w \, dA \right\} dt = 0 \tag{13.12}$$

The variations in Eq. (13.12) can be evaluated using integration by parts as follows:

$$\begin{aligned} I_1 &= \delta \int_{t_1}^{t_2} \frac{P}{2} \iint_A \left(\frac{\partial w}{\partial x} \right)^2 dA \, dt = P \int_{t_1}^{t_2} \iint_A \frac{\partial w}{\partial x} \frac{\partial}{\partial x} (\delta w) \, dA \, dt \\ &= P \int_{t_1}^{t_2} \left[\oint_C \frac{\partial w}{\partial x} \delta w l_x \, dC - \iint_A \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x} \right) \delta w \, dA \right] dt \end{aligned} \tag{13.13}$$

$$I_2 = \delta \int_{t_1}^{t_2} \frac{P}{2} \iint_A \left(\frac{\partial w}{\partial y}\right)^2 dA\,dt = P \int_{t_1}^{t_2} \iint_A \frac{\partial w}{\partial y}\frac{\partial}{\partial y}(\delta w)\,dA\,dt$$

$$= P \int_{t_1}^{t_2} \left[\oint_C \frac{\partial w}{\partial y}\delta w l_y\,dC - \iint_A \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)\delta w\,dA \right] dt \qquad (13.14)$$

$$I_3 = \delta \int_{t_1}^{t_2} \frac{\rho}{2} \iint_A \left(\frac{\partial w}{\partial t}\right)^2 dA\,dt = \frac{\rho}{2} \iint_A \delta \int_{t_1}^{t_2} \left(\frac{\partial w}{\partial t}\right)^2 dA\,dt \qquad (13.15)$$

By using integration by parts with respect to time, the integral I_3 can be written as

$$I_3 = \frac{\rho}{2} \iint_A \delta \int_{t_1}^{t_2} \left(\frac{\partial w}{\partial t}\right)^2 dt\,dA = \rho \iint_A \left[\frac{\partial w}{\partial t}\delta w \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\right)\delta w\,dt \right] dA \qquad (13.16)$$

Since δw vanishes at t_1 and t_2, Eq. (13.16) reduces to

$$I_3 = -\rho \int_{t_1}^{t_2} \iint_A \frac{\partial^2 w}{\partial t^2}\delta w\,dA\,dt \qquad (13.17)$$

$$I_4 = \delta \int_{t_1}^{t_2} \iint_A f w\,dA\,dt = \int_{t_1}^{t_2} \iint_A f \delta w\,dA\,dt \qquad (13.18)$$

Using Eqs. (13.13), (13.14), (13.17), and (13.18), Eq. (13.12) can be expressed as

$$\int_{t_1}^{t_2} \left[-\iint_A \left(P\frac{\partial^2 w}{\partial x^2} + P\frac{\partial^2 w}{\partial y^2} + f - \rho\frac{\partial^2 w}{\partial t^2} \right)\delta w\,dA \right] dt$$
$$+ \int_{t_1}^{t_2} \left[\oint_C P\left(\frac{\partial w}{\partial x}l_x + \frac{\partial w}{\partial y}l_y\right)\delta w\,dC \right] dt = 0 \qquad (13.19)$$

By setting each of the expressions under the brackets in Eq. (13.19) equal to zero, we obtain the differential equation of motion for the transverse vibration of the membrane as

$$P\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + f = \rho\frac{\partial^2 w}{\partial t^2} \qquad (13.20)$$

and the boundary condition as

$$\oint_C P\left(\frac{\partial w}{\partial x}l_x + \frac{\partial w}{\partial y}l_y\right)\delta w\,dC = 0 \qquad (13.21)$$

Note that Eq. (13.21) will be satisfied for any combination of boundary conditions for a rectangular membrane. For a fixed edge:

$$w = 0 \quad \text{and hence} \quad \delta w = 0 \qquad (13.22)$$

For a free edge with $x = 0$ or $x = a$, $l_y = 0$ and $l_x = 1$:

$$P \frac{\partial w}{\partial x} = 0 \tag{13.23}$$

With $y = 0$ or $y = b$, $l_x = 0$ and $l_y = 1$:

$$P \frac{\partial w}{\partial y} = 0 \tag{13.24}$$

For arbitrary geometries of the membrane, Eq. (13.21) can be expressed as

$$\oint_C P \frac{\partial w}{\partial n} \delta w \, dC = 0 \tag{13.25}$$

which will be satisfied when either the edge is fixed with

$$w = 0 \quad \text{and hence} \quad \delta w = 0 \tag{13.26}$$

or the edge is free with

$$P \frac{\partial w}{\partial n} = 0 \tag{13.27}$$