

1. PROBLEM STATEMENT

We want to compute the steady-state oscillations of the equation

$$(1) \quad \begin{aligned} u_{tt} + c^2 u_{xxxx} &= f(x) \cos \Omega t, \\ u(0, t) &= u_{xx}(0, t) = u(L, t) = u_{xx}(L, t) = 0. \end{aligned}$$

2. THE HOMOGENEOUS EQUATION

We first look at the homogeneous equation:

$$\begin{aligned} u_{tt} + c^2 u_{xxxx} &= 0, \\ u(0, t) &= u_{xx}(0, t) = u(L, t) = u_{xx}(L, t) = 0, \end{aligned}$$

and separate the variables by letting $u(x, t) = X(x)T(t)$. Thus,

$$X(x)T''(t) + c^2 X''''(x)T(t) = 0,$$

and therefore

$$-\frac{T''(t)}{T(t)} = c^2 \frac{X''''(x)}{X(x)} = \omega^2,$$

and we arrive at these two ordinary differential equations

$$T''(t) + \omega^2 T(t) = 0, \quad X''''(x) + \left(\frac{\omega}{c}\right)^2 X(x) = 0.$$

For convenience, we define $\lambda = \sqrt{\frac{\omega}{c}}$, whereby the second equation takes the form $X'''' + \lambda^4 X = 0$. Then we solve for $T(t)$ and $X(x)$:

$$\begin{aligned} T(t) &= A_n \cos \omega t + B_n \sin \omega t, \\ X(x) &= c_1 \sin \lambda x + c_2 \cos \lambda x + c_3 \sinh \lambda x + c_4 \cosh \lambda x. \end{aligned}$$

The original PDE's boundary conditions translate to $X(0) = X''(0) = X(L) = X''(L) = 0$. Considering that

$$X''(x) = \lambda^2 [-c_1 \sin \lambda x - c_2 \cos \lambda x + c_3 \sinh \lambda x + c_4 \cosh \lambda x],$$

we see that $X(0) = X''(0) = 0$ imply that $c_2 = c_4 = 0$, and thus we are left with $X(x) = c_1 \sin \lambda x + c_3 \sinh \lambda x$. Applying the boundary conditions $X(L) = X''(L) = 0$ yields

$$\begin{aligned} c_1 \sin \lambda L + c_3 \sinh \lambda L &= 0, \\ -c_1 \sin \lambda L + c_3 \sinh \lambda L &= 0. \end{aligned}$$

Adding the two equations we get $2c_3 \sinh \lambda L = 0$. We don't want $\lambda = 0$ since that gives us the trivial solution only. But if $\lambda \neq 0$, we have $\sinh \lambda L \neq 0$, and therefore $c_3 = 0$. Then either of the two equations above imply that $c_1 \sin \lambda L = 0$.

We don't want $c_1 = 0$ since that gives us the trivial solution only. Therefore $\sin \lambda L = 0$ which has infinitely many solutions

$$\lambda_n = \frac{n\pi}{L}, \quad n = 1, 2, \dots$$

Referring to the definition $\lambda = \sqrt{\frac{\omega}{c}}$, we denote the corresponding ω values by $\omega_n = c\lambda_n^2$.

The family of function $\{\sin \lambda_n x\}_{n=1}^{\infty}$ is an orthogonal basis for square-integrable functions on $(0, L)$ under the inner product

$$(f, g) = \int_0^L f(x)g(x) dx.$$

Therefore, any square-integrable function $f(x)$ may be expressed as

$$(2) \quad f(x) = \sum_{n=1}^{\infty} a_n \sin \lambda_n x,$$

where

$$(3) \quad a_n = \frac{2}{L} \int_0^L f(x) \sin \lambda_n x dx.$$

3. STEADY-STATE OSCILLATIONS

We return to the original statement of the problem in (1). We expand $f(x)$ into a sum of eigenfunctions as in (2), where the coefficients a_n may be computed through (3). We also expand the solution $u(x, t)$ into a sum of eigenfunctions

$$u(x, t) = \sum_{n=1}^{\infty} p_n(t) \sin \lambda_n x,$$

where the coefficients $p_n(t)$ are to be determined. Plug these into the PDE in (1). We get:

$$\sum_{n=1}^{\infty} p_n''(t) \sin \lambda_n x + c^2 \sum_{n=1}^{\infty} \lambda_n^4 p_n(t) \sin \lambda_n x = \left(\sum_{n=1}^{\infty} a_n \sin \lambda_n x \right) \cos \Omega t,$$

whence

$$p_n''(t) + c^2 \lambda_n^4 p_n(t) = a_n \cos \Omega t.$$

Recalling the definition of ω_n , this takes the form

$$p_n''(t) + \omega_n^2 p_n(t) = a_n \cos \Omega t.$$

Look for a particular solution of this ODE in the form $p_n(t) = K_n \cos \Omega t$. Plugging that form into the ODE we get

$$-\Omega^2 K_n + \omega_n^2 K_n = a_n,$$

whence

$$K_n = \frac{a_n}{\omega_n^2 - \Omega^2}.$$

We conclude that the steady-state oscillations of (1) are given by

$$(4) \quad u(x, t) = \left(\sum_{n=1}^{\infty} \frac{a_n}{\omega_n^2 - \Omega^2} \sin \lambda_n x \right) \cos \Omega t.$$

4. THE SPECIAL FORCING FUNCTION

The expression (4) gives the steady-state oscillations of the beam in (1) for any function $f(x)$. In particular if f is given as

$$f(x) = F_0 \left(1 - \frac{x}{L}\right),$$

then from (3) we get

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \sin \lambda_n x \, dx \\ &= \frac{2F_0}{L} \int_0^L \left(1 - \frac{x}{L}\right) \sin \lambda_n x \, dx \\ &= \left(\frac{2F_0}{L}\right) \left(\frac{L}{n\pi}\right) = \frac{2F_0}{n\pi}, \end{aligned}$$

and therefore

$$u(x, t) = \frac{2F_0}{\pi} \left(\sum_{n=1}^{\infty} \frac{1}{n(\omega_n^2 - \Omega^2)} \sin \lambda_n x \right) \cos \Omega t.$$