Generalization of the Abel inverse Riccati(AIR) equation via trigonometric function

Huang Shaoxuan, Steven

Beijing Normal University-HongKong Baptist university united college, Jintong Road 2000, Zhuhai, Canton, China

Abstract. The integrability of the Abel differential equation is an unsolved problem since it was mentioned by the great Norway mathematician Niels Henrik Abel. Although there is not general method to tell whether a random given Abel equation is integrable or not, several solvable classes has been mentioned over the past 30 years. In this work, we consider a special non-Liouville integrable abel equation class which is called Abel inverse Riccati(AIR) equation. Using a result mentioned by Liouville, we rewrite the AIR equation in trigonometric form. And we also obtain some similar trigonometric form of this class of Abel equation. Then, using a technique to transform the trigonometry functions into algebraic functions, we obtain the following results:1, we conclude that a more general form of algebraic Abel equation is not Liouville integrable. 2, we obtain some new rational differential equation which has non-Liouville integral. All our computation is based on the computer algebra system(CAS) Maple 2021/22.

Keywords: Abel differential equation · rational differential equation · Liouville integrability · Non-elementary solution

1 Introduction

The general Abel differential equation of the first kind has the form:

$$\frac{dy}{dx} = f_1(x)y^3 + f_2(x)y^2 + f_3(x)y + f_4(x) \tag{1}$$

Where $f_1(x) \neq 0$. The Abel differential equation of the second kind has the form:

$$(g_1(x)y + g_2(x))\frac{dy}{dx} = f_1(x)y^3 + f_2(x)y^2 + f_3(x)y + f_4(x) \tag{2}$$

Where $f_1(x) \neq 0$, $g_1(x) \neq 0$. It is shown that, by change of variables:

$$\frac{g_1(x)}{u} = g_1(x)y + g_2(x) \tag{3}$$

That the abel equation of the second kind is reduced to the first kind. So the Abel equation of the second kind is actually equivalent to the first kind. There have been many works on the Liouville integrable case of the Abel equation, readers may refer to [6],[7],[4]

1.1 Preliminary

The Abel Inverse Riccati equation, first derived in [3], is an Abel equation of the second kind of the form:

$$\frac{dy}{dx} = \frac{a_1y^3 + a_2y^2 + a_3y + a_4}{(b_1x^2 + b_2x + b_3)y + c_1x^2 + c_2x + c_3} \tag{4}$$

As the name indicates, by inverse variables $x \to y, y \to x$, this equation becomes a Riccati equation

$$\frac{dy}{dx} = \frac{(b_1x + c_1)}{(a_1x^3 + a_2x^2 + a_3x + a_4)}y^2 + \frac{(b_2x + c_2)}{(a_1x^3 + a_2x^2 + a_3x + a_4)}y + \frac{(b_3x + c_3)}{(a_1x^3 + a_2x^2 + a_3x + a_4)} \tag{5}$$

This characteristics will be used for our generalization in section 2 and 3. The main idea for solving the AIR equation is to divide it into classes. Rewrite the AIR equation into the form:

$$\frac{dy}{dx} = \frac{(y-\rho_1)(y-\rho_2)(y-\rho_3)}{(b_1x^2 + b_2x + b_3)y + c_1x^2 + c_2x + c_3} \tag{6}$$

class 1 If there is three distincts roots ρ_i, then, according to [5] by applying certain Mobius transformation[10]:

$$y = \frac{px + q}{rs + t} \tag{7}$$

The AIR equation is transformed to the form:

$$\frac{dy}{dx} = \frac{y(y-1)}{(b_1x^2+b_2x+b_3)y+(c_1x^2+c_2x+c_3)} \tag{8}$$

for some new constants b_i, c_i, Applying change of variables:

$$x \to \frac{x(1-x)\frac{dy}{dx}}{(b_1x+c_1)y}, y \to x \tag{9}$$

class 1 is reduced to the second order Heun differential equation:

$$\frac{d^2y}{dx^2} = \frac{b_1(b_2-1)x^2+((b_2-1)c_1+b_1c_2)x+c_1(1+c_2)}{x(b_1x+c_1)(x-1)}\frac{dy}{dx} - \frac{(b_3x+c_3)(b_1x+c_1)}{x^2(x-1)^2}y \tag{10}$$

which has the non-Liouville solution of HeunG function.

class 2 If there is two distincts roots ρ_i, then by applying certain Mobius transformation, the AIR equation is transformed to the form:

$$\frac{dy}{dx} = \frac{y}{(b_1x^2+b_2x+b_3)y+(c_1x^2+c_2x+c_3)} \tag{11}$$

for some new constants b_i, c_t. Inversing the variables x and y, and then transform the Riccati equation to the linear second order equation by change of variables[6]:

$$y \to -\frac{xy}{\frac{dy}{dx}(b_1x+c_1)} \tag{12}$$

we arrive at:

$$\frac{d^2y}{dx^2} = \frac{b_1b_2x^2+(b_1c_2+b_2c_1)x-b_1+b_1b_2}{b_1x+c_1}\frac{dy}{dx} - \frac{(b_1b_3x^2+(b_1c_3+b_3c_1)x+c_1c_3)}{x^2}y \tag{13}$$

If we rewrite this equation into normal form, it becomes the confluent Heun equation[2], the non-Liouville solution of which can be expressed via the HeunC function.

class3 If there is one distinct roots ρ_i, then by applying certain Mobius transformation, the AIR equation is transformed to the form:

$$\frac{dy}{dx} = \frac{1}{(b_1x^2 + b_2x + b_3)y + (c_1x^2 + c_2x + c_3)} \tag{14}$$

Using similar transformation that we used in solving class 2, we will arrive at:

$$\frac{d^2y}{dx^2} = \frac{b_1b_2x^2 + (b_1c_2 + b_2c_1)x + b_1 + c_1c_2}{b_1x + c_1}\frac{dy}{dx} - (b_1b_3x^2 + (b_1c_3 + b_3c_1)x + c_1c_3)y \tag{15}$$

Rewriting it to the normal form results in biconfluent Heun equation[2].

2 Generalization process

In this section we consider the following two Abel differential equations of the second kind. The first equation:

$$(a_1y^3 + a_2y^2 + a_3y + a_4)A(x)^n = \frac{dy}{dx}(P(x)y + Q(x)) \tag{16}$$

where a_i are arbitrary constants, $n > 0$, $P(x), Q(x)$ are non-fractional algebraic expressions[1](does not contain negative power) in terms of x. $A(x)$ is two degree polynomial which has two distinct roots ρ_1, ρ_2. The second equation:

$$(a_1y^3 + a_2y^2 + a_3y + a_4) = \frac{dy}{dx}((b_1U(x) + b_2T(x) + b_3R(x))y + c_1U(x) + c_2T(x) + c_3R(x)) \tag{17}$$

where a_i, b_i, c_i are arbitrary constants, and

$$U(x) = P(x)Q(x)^{-1}A(x) \tag{18}$$

$$T(x) = A(x)^{-n+1}Q(x) \tag{19}$$

[1] Here we use an informal definition. We also consider function with the variables raising to a irrational number power and complex number coefficients as algebraic. We change the definition here for simplifying our discussions.

$$R(x) = A(x)^{n+1}Q(x)^{-1} \tag{20}$$

$n > 0, P(x), Q(x)$ are non-fractional algebraic expressions. $A(x)$ is two degree polynomial which has two distinct roots ρ_1, ρ_2. In the above two Abel equations, $P(x), Q(x)$ do not contain the root ρ_1, ρ_2 and a_i do not equal to zero simultaneously. Generally, it's difficult to tell whether the above two equations are Liouville integrable or not. However, by constructing a counter example using the AIR equation, we can claim that, in general, for arbitrarily chosen constants $a_i, A(x), n$ in the above equations, there exists some non-Liouville integrable cases.

2.1 trigonometric forms of AIR equation

It is shown by Liouville[6], that equation of the form:

$$\frac{dy}{dx} = f(x)cos(y) + g(x)sin(y) + h(x) \tag{21}$$

can be transformed to an arbitrary Riccati equation by the tangent half-angle substitution $y \to 2arctan(y)$[11]. Based on this idea, we apply the inverse substitution to the equation 5, to get:

$$\frac{dy}{dx} = \frac{(b_1x + c_1)}{(a_1x^3 + a_2x^2 + a_3x + a_4)}cos(y) + \frac{(b_2x + c_2)}{(a_1x^3 + a_2x^2 + a_3x + a_4)}sin(y) + \frac{(b_3x + c_3)}{(a_1x^3 + a_2x^2 + a_3x + a_4)} \tag{22}$$

for some new constants a_i, b_i, c_i And inverse the variables x and y, we obtain the trigonometric forms of the AIR equation:

$$((b_1y + c_1)cos(x) + (b_2y + c_2)sin(x) + b_3y + c_3)\frac{dy}{dx} = a_1y^3 + a_2y^2 + a_3y + a_4 \tag{23}$$

Here we point out that, despite from Liouville's results, there are some other homogeneous trigonometric form of the AIR equation:

$$((b_1y + c_1)cos(x) + (b_2y + c_2)sec(x) + (b_3y + c_3)tan(x))\frac{dy}{dx} = a_1y^3 + a_2y^2 + a_3y + a_4 \tag{24}$$

The above equation can be reduced to the AIR equation by applying transformation $x \rightarrow arcsin(x)$

$$((b_1y+c_1)sin(x)+(b_2y+c_2)csc(x)+(b_3y+c_3)cot(x))\frac{dy}{dx}=a_1y^3+a_2y^2+a_3y+a_4 \tag{25}$$

The above equation can be reduced to the AIR equation by applying transformation $x \rightarrow arccos(x)$

2.2 multiple angle tangent substitution

In order to convert the trigonometric form to the algebraic form, the well known tangent half angle substitution is introduced. However, we notice that despite from the half angle substitution, the transformation:

$$x \rightarrow n * arctan(x), n > 0 \tag{26}$$

will also transform the equations (23),(24),(25) into algebraic form. Let us consider the exact form after such kind of substitution. For the convenience of calculation, set $n = 2n$, then, according to Euler formula and basic properties of the trigonometric functions:

$$sin(2n\theta)=(e^{i2n\theta}-e^{-i2n\theta})/2i=\frac{((cos(\theta)+isin(\theta))^{2n}-(cos(\theta)-isin(\theta))^{2n})}{2i*(cos^2(\theta)+sin^2(\theta))^n} \tag{27}$$

$$cos(2n\theta)=(e^{i2n\theta}+e^{-i2n\theta})/2=\frac{((cos(\theta)+isin(\theta))^{2n}+(cos(\theta)-isin(\theta))^{2n})}{2(cos^2(\theta)+sin^2(\theta))^n} \tag{28}$$

$$tan(2n\theta)=\frac{(cos(\theta)+isin(\theta))^{2n}-(cos(\theta)-isin(\theta))^{2n}}{i*((cos(\theta)+isin(\theta))^{2n}+(cos(\theta)-isin(\theta))^{2n})} \tag{29}$$

Divide $cos^{2n}(\theta)$ on the denominator and divisor, and substitute $\theta = arctan(x)$, we obtain:

$$sin(2n*arctan(x))=\frac{(1+i*x)^{2n}-(1-i*x)^{2n}}{2i(1+x^2)^n} \tag{30}$$

$$cos(2n*arctan(x))=\frac{(1+i*x)^{2n}+(1-i*x)^{2n}}{2(1+x^2)^n} \tag{31}$$

$$tan(2n*arctan(x))=\frac{(1+i*x)^{2n}-(1-i*x)^{2n}}{i((1+i*x)^{2n}+(1-i*x)^{2n})} \tag{32}$$

So the equation (23) after the substitution becomes:

$$(a_1y^3+a_2y^2+a_3y+a_4)(x^2+1)^{n-1} = \frac{dy}{dx}((b_1y+c_1)(1-i*x)^{2n}+(b_2y+c_2)(1+i*x)^{2n}+(b_3y+c_3)(x^2+1)^n) \tag{33}$$

for some new constants a_i, b_i, c_i. This equation is exactly of the form (16). Since it's generated from the AIR equation, it does not have Liouville integral. To transform it back to AIR, reverse the multiple angle tangent substitution, and apply the tangent half angle substitution. the equation (24) after substitution becomes:

$$(a_1y^3 + a_2y^2 + a_3y + a_4) = \frac{dy}{dx}((b_1y + c_1)T(x) + (b_2y + c_2)R(x) + (b_3y + c_3)U(x))(x^2 + 1) \tag{34}$$

for some new constants a_i, b_i, c_i, where:

$$T(x) = \frac{(1 + i * x)^{2n} - (1 - i * x)^{2n}}{(1 + i * x)^{2n} + (1 - i * x)^{2n}}, R(x) = \frac{(1 + i * x)^{2n} + (1 - i * x)^{2n}}{(x^2 + 1)^n}, U(x) = R(x)^{-1} \tag{35}$$

The equation (25) after substitution becomes:

$$(a_1y^3 + a_2y^2 + a_3y + a_4) = \frac{dy}{dx}((b_1y + c_1)T(x) + (b_2y + c_2)R(x) + (b_3y + c_3)U(x))(x^2 + 1) \tag{36}$$

for some new constants a_i, b_i, c_i, where:

$$T(x) = \frac{(1 + i * x)^{2n} + (1 - i * x)^{2n}}{(1 + i * x)^{2n} - (1 - i * x)^{2n}}, R(x) = \frac{(1 + i * x)^{2n} - (1 - i * x)^{2n}}{(x^2 + 1)^n}, U(x) = R(x)^{-1} \tag{37}$$

Both (34) and (36) belong to the form of (17). To transform it back to AIR equation, inverse the multiple angle tangent substitution, and then apply the sine/cosine transformation. Based on the (33),(34) and (36), by applying scaling transformation $x \to \alpha * x$ and shift transformation $x \to x + \beta$, we will obtain a more general form. For (33) it becomes:

$$(a_1y^3+a_2y^2+a_3y+a_4)[(x-\rho_1)(x-\rho_2)]^{n-1} = \frac{dy}{dx}((b_1y+c_1)(x-\rho_1)^{2n}+(b_2y+c_2)(x-\rho_2)^{2n}+(b_3x+c_3)[(x-\rho_1)(x-\rho_2 \tag{38}$$

where a_i, b_i, c_i, ρ_i can be arbitrary constants. So we are able to claim for any $a_i, A(x), n$ in (16), there exists some cases which are not Liouville integrable. For (34),(36) it becomes:

$$(a_1y^3+a_2y^2+a_3y+a_4) = \frac{dy}{dx}((b_1y+c_1)T(x)+(b_2y+c_2)R(x)+(b_3y+c_3)U(x))(x-\rho_1)(x-\rho_2) \tag{39}$$

where a_i, b_i, c_i, ρ_i can be arbitrary constants, and :

$$T(x) = \frac{(x-\rho_1)^{2n}+(x-\rho_2)^{2n}}{(x-\rho_1)^{2n}-(x-\rho_2)^{2n}}, R(x) = \frac{(x-\rho_1)^{2n}-(x-\rho_2)^{2n}}{[(x-\rho_1)(x-\rho_2)]^n}, U(x) = R(x)^{-1} \tag{40}$$

for (34),

$$T(x) = \frac{(x-\rho_1)^{2n}-(x-\rho_2)^{2n}}{(x-\rho_1)^{2n}+(x-\rho_2)^{2n}}, R(x) = \frac{(x-\rho_1)^{2n}+(x-\rho_2)^{2n}}{[(x-\rho_1)(x-\rho_2)]^n}, U(x) = R(x)^{-1} \tag{41}$$

for (36). It is of the form of (17), so we can claim that for arbitrary chosen $a_i, A(x), n$ in (17) there are some non-Liouville integrable cases. We can verify our results using Maple. Although Maple 2022 cannot due with the general case of (33),(34) and (36), by taking some specific examples, and trace the solution steps by setting the **infolevel** parameter, we can observe which Abel equation class it falls into.

Fig. 1. setting the parameter

$$dsolve\left(eval\left(u(t)\cdot\left(t^2+1\right)^{n-1}=\frac{d}{dt}u(t)\cdot\left(u(t)\cdot\left((-1+I\cdot t)^2\right)^n+\left((I\cdot t+1)^2\right)^n\right),\ n=3\right)\right)$$

Fig. 2. an example

```
   <- special function solution successful
<- Riccati to 2nd Order successful
The transformation which yields the equation in AIR form has inverse: {t = 12*I/(t+I)^3*(-t+I)^3, u(t) =
144*u(t)}
<- Abel class AIR 3-parameter reducible to Riccati successful
<- Abel successful
```

$$\_C1+\frac{\mathrm{BesselI}\left(1,\sqrt{144}\sqrt{u(t)}\right)\sqrt{144}\sqrt{u(t)}+\frac{12\ \mathrm{I}\ \mathrm{BesselI}\left(0,\sqrt{144}\sqrt{u(t)}\right)(-t+\mathrm{I})^3}{(t+\mathrm{I})^3}}{\sqrt{144}\sqrt{u(t)}\ \mathrm{BesselK}\left(1,\sqrt{144}\sqrt{u(t)}\right)-\frac{12\ \mathrm{I}\ \mathrm{BesselK}\left(0,\sqrt{144}\sqrt{u(t)}\right)(-t+\mathrm{I})^3}{(t+\mathrm{I})^3}}=0 \quad (23)$$

Fig. 3. tracing solution steps

Remark

1. When b_1, c_1 or b_2, c_2 equal to 0 simultaneously, equation (33) become Liouville integrable since the Riccati equation it corresponds to becomes Bernoulli equation or linear equation. When b_2, c_2 equal to 0 simultaneously, equation (34),(36) become Liouville integrable.

2. The hyperbolic function $sinh(x), cosh(x), tanh(x)$ have similar properties to the trigonometric functions. In fact, applying multiple angle hyperbolic tangent substitution to the hyperbolic forms of AIR equation:

$$((b_1y+c_1)cosh(x)+(b_2y+c_2)sinh(x)+b_3y+c_3)\frac{dy}{dx}=a_1y^3+a_2y^2+a_3y+a_4 \quad (42)$$

$$((b_1y+c_1)cosh(x)+(b_2y+c_2)sech(x)+(b_3y+c_3)tanh(x))\frac{dy}{dx}=a_1y^3+a_2y^2+a_3y+a_4 \quad (43)$$

$$((b_1y+c_1)sinh(x)+(b_2y+c_2)csch(x)+(b_3y+c_3)coth(x))\frac{dy}{dx}=a_1y^3+a_2y^2+a_3y+a_4 \quad (44)$$

will produce similar result to the above.

3 Integrable rational equation

The rational first order differential equation, which also called second order autonomous polynomial system, is of the form:

$$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)} \tag{45}$$

where $M(x,y), N(x,y)$ are polynomial of two variables x,y. In [7], Keying Guan dicussed about the necessary and sufficient condition of the Liouville integrability of a given system. In [1],[8],[9], several authors consider in what conditions can a certain type of system be reduced to Abel equation or Riccati equation. In this work, we consider a special case of the equation (23):

$$((b_1y + c_1)cos(x) + (b_2y + c_2)sin(x) + b_3y + c_3)\frac{dy}{dx} = a_1y^3 + a_1y + a_2y^2 + a_2 \tag{46}$$

This equation can be further converted to the trigonometry form:

$$((b_1tan(y) + c_1)cos(x) + (b_2tan(y) + c_2)sin(x) + b_3tan(y) + c_3)\frac{dy}{dx} = a_1tan(y) + a_2 \tag{47}$$

By change of variables $y \to tan(y)$. similarly, for (24) and (25), if $a_1 = a_3, a_2 = a_4$, then we can transform them to:

$$((b_1tan(y) + c_1)cos(x) + (b_2tan(y) + c_2)sec(x) + (b_3tan(y) + c_3)tan(x))\frac{dy}{dx} = a_1tan(y) + a_2 \tag{48}$$

$$((b_1tan(y) + c_1)sin(x) + (b_2tan(y) + c_2)csc(x) + (b_3tan(y) + c_3)cot(x))\frac{dy}{dx} = a_1tan(y) + a_2 \tag{49}$$

respectively. For the above equation, apply multiple tangent substitution to y,x:

$$y \to 2m * arctan(y), x \to 2n * arctan(x), m, n \in N \tag{50}$$

we will obtain some new integrable rational differential equations. To speak specific, equation (47) becomes:

$$((b_1R_1(y)+c_1)T_1(x)+(b_2R_1(y)+c_2)T_2(x)+b_3R_1(y)+c_3)(x^2+1)\frac{dy}{dx}=(a_1R_1(y)+a_2)(y^2+1) \quad (51)$$

for some new constants a_i, b_i, c_i. Equation (48) becomes:

$$((b_1R_1(y)+c_1)T_1(x)+(b_2R_1(y)+c_2)T_1(x)^{-1}+(b_3R_1(y)+c_3)T_3(x))(x^2+1)\frac{dy}{dx}=(a_1R_1(y)+a_2)(y^2+1) \quad (52)$$

for some new constants a_i, b_i, c_i. Equation (49) becomes:

$$((b_1R_1(y)+c_1)T_2(x)+(b_2R_1(y)+c_2)T_2(x)^{-1}+(b_3R_1(y)+c_3)T_3(x)^{-1})(x^2+1)\frac{dy}{dx}=(a_1R_1(y)+a_2)(y^2+1) \quad (53)$$

for some new constants. In the above equation:

$$T_1(x)=\frac{(1+i*x)^{2n}+(1-i*x)^{2n}}{(1+x^2)^n} \quad (54)$$

$$T_2(x)=\frac{(1+i*x)^{2n}-(1-i*x)^{2n}}{(1+x^2)^n} \quad (55)$$

$$T_3(x)=\frac{(1+i*x)^{2n}-(1-i*x)^{2n}}{(1+i*x)^{2n}+(1-i*x)^{2n}} \quad (56)$$

$$R_1(y)=\frac{(1+i*y)^{2m}-(1-i*y)^{2m}}{(1+i*y)^{2m}+(1-i*y)^{2m}} \quad (57)$$

Like mentioned before, for (47),(48) and (49), changing the trigonometric functions in terms of x or y into the corresponding hyperbolic function, it can still be transformed back to the AIR equation. We apply the multiple angle tangent or hyperbolic tangent substitution according to the function type, and apply scaling transformation and shift transformation on x and y. Consequently, we will arrive at a more general result.

$$((b_1R_1(y)+c_1)T_1(x)+(b_2R_1(y)+c_2)T_2(x)+b_3R_1(y)+c_3)(x-\rho_1)(x-\rho_2)\frac{dy}{dx} = (a_1R_1(y)+a_2)(y-\alpha_1)(y-\alpha_2) \tag{58}$$

$$((b_1R_1(y)+c_1)T_1(x)+(b_2R_1(y)+c_2)T_1(x)^{-1}+(b_3R_1(y)+c_3)T_3(x))(x-\rho_1)(x-\rho_2)\frac{dy}{dx} = (a_1R_1(y)+a_2)(y-\alpha_1)(y- \tag{59}$$

$$((b_1R_1(y)+c_1)T_2(x)+(b_2R_1(y)+c_2)T_2(x)^{-1}+(b_3R_1(y)+c_3)T_3(x)^{-1})(x-\rho_1)(x-\rho_2)\frac{dy}{dx} = (a_1R_1(y)+a_2)(y-\alpha_1)(y \tag{60}$$

where:

$$T_1(x) = \frac{(x-\rho_1)^{2n} + (x-\rho_2)^{2n}}{[(x-\rho_1)(x-\rho_2)]^n} \tag{61}$$

$$T_2(x) = \frac{(x-\rho_1)^{2n} - (x-\rho_2)^{2n}}{[(x-\rho_1)(x-\rho_2)]^n} \tag{62}$$

$$T_3(x) = \frac{(x-\rho_1)^{2n} - (x-\rho_2)^{2n}}{(x-\rho_1)^{2n} + (x-\rho_2)^{2n}} \tag{63}$$

$$R_1(y) = \frac{(y-\alpha_1)^{2m} - (y-\alpha_2)^{2m}}{(y-\alpha_1)^{2m} + (y-\alpha_2)^{2m}} \tag{64}$$

the $a_i, b_i, c_i, \rho_i, \alpha_i$ can be arbitrary constants. Both the general cases and most special cases of (58),(59) and (60) are unsolvable in maple.

4 Conclusion

In this work, based on the AIR equation, by rewriting it to trigonometric form and apply some substitution to transfer it back to algebraic form, we obtain some new integrable Abel equation and rational equation.

References

1. Álvarez, A., Bravo, J.L., Sánchez, F.: Planar systems and abel equations. Communications on Pure and Applied Analysis **21**(10), 3463–3478 (2022)

2. Arscott, F.M., Slavyanov, S.Y., Schmidt, D., Wolf, G., Maroni, P., Duval, A.: Heun's Differential Equations. Clarendon Press (1995)
3. Cheb-Terrab, E.S.: A connection between abel and hypergeometric differential equations. European Journal of Applied Mathematics **16**(1), 53–63 (2005)
4. Cheb-Terrab, E.S., Roche, A.D.: An abel ordinary differential equation class generalizing known integrable classes. European Journal of Applied Mathematics **14**(2), 217–229 (2003)
5. Cheb-Terrab, E.: Solutions for the general, confluent and biconfluent heun equations and their connection with abel equations. Journal of Physics A: Mathematical and General **37**(42), 9923 (2004)
6. Kamke, E.: Differentialgleichungen Lösungsmethoden und Lösungen: II. Partielle Differentialgleichungen Erster Ordnung für eine Gesuchte Funktion. Springer-Verlag (2013)
7. Keying, G., Jinzhi, L.: Integrability of second order autonomous system. Annals of Differential Equations **18**(2), 117–135 (2002)
8. Nicklason, G.R.: An abel type cubic system. Electronic Journal of Differential Equations **2015**(189), 1–17 (2015)
9. Nicklason, G.: Homogeneous-like generalized cubic systems. International Journal of Differential Equations **2016** (2016)
10. Stein, E.M., Shakarchi, R.: Complex analysis, vol. 2. Princeton University Press (2010)
11. Stewart, J.: Calculus. Cengage Learning (2015)