```
      # normalisation
      u:=w/LinearAlgebra[VectorNorm](w,Euclidean);
      # iteration
      w:=A.u;
      # eigenvalue estimate
      lambda:=w.u;
    end do;
    # output
    return lambda,u;
  end proc:
```

c) Deflation

So far: given $\underline{\underline{A}}$ with $|\lambda^{(1)}| > |\lambda^{(2)}| > \ldots > |\lambda^{(n)}|$ we can compute $\lambda^{(1)}$ and $\underline{u}^{(1)}$. How to obtain the other eigenvalues $\lambda^{(2)}$, $\lambda^{(3)}$,..., $\lambda^{(n)}$?

Idea: Construct a matrix $\underline{\underline{B}}$ with eigenvalues $0, \lambda^{(2)}, \lambda^{(3)}, \ldots, \lambda^{(n)}$ ("deflate" the matrix $\underline{\underline{A}}$, remove $\lambda^{(1)}$). Then $\lambda^{(2)}$ can be obtained by the power method.

Example 4.3: $\underline{\underline{A}} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, *eigenvalues and eigenvectors*

$$\lambda^{(1)} = 3, \quad \underline{u}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{("leading")}, \qquad \lambda^{(1)} = 2, \quad \underline{u}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underline{\underline{B}} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} - \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}}_{\begin{pmatrix} 1 \\ 0 \end{pmatrix} (3,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

has eigenvalues 0 *and* 2*. If* $\underline{a}^T = (3,0)$ *denotes the* $1st$ *row of* $\underline{\underline{A}}$ *(with the 1st component of* $\underline{u}^{(1)}$ *being* 1*) we have* $\underline{\underline{B}} = \underline{\underline{A}} - \underline{u}^{(1)}\underline{a}^T$*. How to generalise this example?*

Construction of $\underline{\underline{B}}$:

- $\underline{\underline{A}}$ given, $\lambda^{(1)}$ and $\underline{u}^{(1)}$ known (e.g., by power method).

- Choose a nonzero component of $\underline{u}^{(1)}$, say $u_p^{(1)}$, (fixes p !).
- Denote by $\underline{a}^T$ the pth row of $\underline{\underline{A}}$, i.e., $\underline{a}^T = (A_{p1}, A_{p2}, \ldots, A_{pn})$. Because of the eigenvalue equation $\underline{\underline{A}}\underline{u}^{(k)} = \lambda^{(k)}\underline{u}^{(k)}$ we have

$$\lambda^{(k)} u_p^{(k)} = \left(\underline{\underline{A}}\,\underline{u}^{(k)}\right)_p = \underline{a}^T \underline{u}^{(k)} .$$

- Consider

$$\underline{\underline{B}} = \underline{\underline{A}} - \frac{1}{u_p^{(1)}}\underline{u}^{(1)}\underline{a}^T, \quad \text{i.e.,} \quad B_{n\ell} = A_{n\ell} - \frac{1}{u_p^{(1)}}\underline{u}^{(1)} u_n^{(1)} A_{p\ell} .$$

$\underline{\underline{B}}$ has the required properties. Namely:

$$\underline{\underline{B}}\underline{u}^{(1)} = \underbrace{\underline{\underline{A}}\,\underline{u}^{(1)}}_{\lambda^{(1)}\underline{u}^{(1)}} - \frac{1}{u_p^{(1)}}\underline{u}^{(1)} \underbrace{\left(\underline{a}^T\underline{u}^{(1)}\right)}_{\lambda^{(1)}u_p^{(1)}} = 0$$

i.e., 0 is eigenvalue of $\underline{\underline{B}}$ (instead of $\lambda^{(1)}$), and for $k = 2, 3, \ldots, n$

$$\underline{\underline{B}}\underline{u}^{(k)} = \underbrace{\underline{\underline{A}}\,\underline{u}^{(k)}}_{\lambda^{(k)}\underline{u}^{(k)}} - \frac{1}{u_p^{(1)}}\underline{u}^{(1)} \underbrace{\left(\underline{a}^T\underline{u}^{(k)}\right)}_{\lambda^{(k)}u_p^{(k)}} = \lambda^{(k)}\left(\underline{u}^{(k)} - \frac{u_p^{(k)}}{u_p^{(1)}}\underline{u}^{(1)}\right)$$

that means

$$\underline{\underline{B}}\left(\underline{u}^{(k)} - \frac{u_p^{(k)}}{u_p^{(1)}}\underline{u}^{(1)}\right) = \lambda^{(k)}\left(\underline{u}^{(k)} - \frac{u_p^{(k)}}{u_p^{(1)}}\underline{u}^{(1)}\right) .$$

Thus $\lambda^{(k)}$, $k = 2, 3, \ldots, n$ are still eigenvalues of $\underline{\underline{B}}$ (but the eigenvectors have changed).

Example 4.4:

$$\underline{\underline{A}} = \begin{pmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

Eigenvalues and eigenvectors

$$\lambda^{(1)} = 6, \quad \underline{u}^{(1)} = \begin{pmatrix} -4 \\ -20/7 \\ 1 \end{pmatrix}, \quad \lambda^{(2)} = 3, \quad \underline{u}^{(2)} = \begin{pmatrix} -1 \\ -1/2 \\ 1 \end{pmatrix}, \quad \lambda^{(3)} = 2, \quad \underline{u}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Deflate $\underline{\underline{A}}$*: choose, e.g.,* $p = 1$

$$u_{p=1}^{(1)} = -4, \quad \underline{a}^T = (-4, 14, 0)$$

$$\underline{\underline{B}} = \begin{pmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{pmatrix} - \frac{1}{(-4)} \begin{pmatrix} -4 \\ -20/7 \\ 1 \end{pmatrix} (-4, 14, 0)$$
$$= \begin{pmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 16 & -56 & 0 \\ 80/7 & -40 & 0 \\ -4 & 14 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ -15/7 & 3 & 0 \\ -2 & 7/2 & 2 \end{pmatrix}$$

Eigenvalues and eigenvectors of $\underline{\underline{B}}$

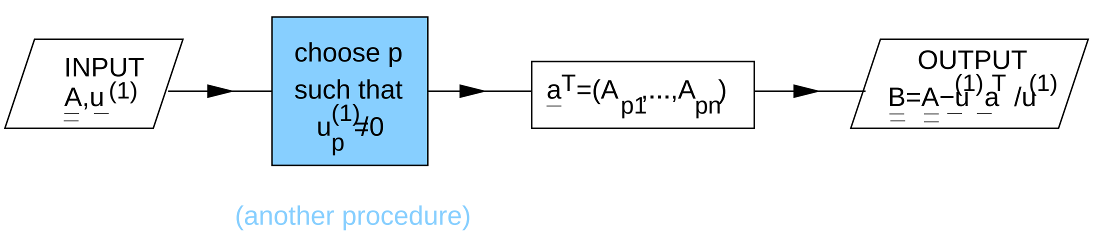
$$3, \begin{pmatrix} 0 \\ 2/7 \\ 1 \end{pmatrix}, \left[2, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad 0, \begin{pmatrix} -4 \\ -20/7 \\ 1 \end{pmatrix} \quad (\text{"non leading"}) \right]$$

Deflate again to make the next eigenvalue the leading one: now $\lambda^{(1)} = 3$, $\underline{u}^{(1)} = \begin{pmatrix} 0 \\ 2/7 \\ 1 \end{pmatrix}$.

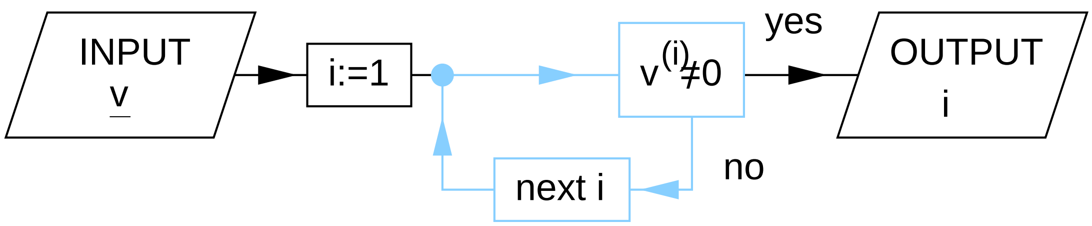
Choose e.g., $p = 3$, *then* $u_p^{(1)} = 1$, $\underline{a}^T = (-2, 2/7, 2)$ *and*

$$\underline{\underline{B}} = \begin{pmatrix} 0 & 0 & 0 \\ -15/7 & 3 & 0 \\ -2 & 7/2 & 2 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 0 \\ 2/7 \\ 1 \end{pmatrix} (-2, 7/2, 2) = \begin{pmatrix} 0 & 0 & 0 \\ -11/2 & 2 & 4/7 \\ 0 & 0 & 0 \end{pmatrix}.$$

Flow chart (for a single deflation step):



"choose p such that $u_p^{(1)} \neq 0$" is another procedure:



Maple code: i) procedure to compute p with $u_p^{(1)} \neq 0$:

```
pindex:=proc(v)
   local i,ndim;
   # number of components
   ndim:=LinearAlgebra[Dimension](v);
   # loop over all components
   for i from 1 to ndim do
      if abs(v[i])>0 then
         # nonzero component
         return i;
      end if;
   end do;
end proc:
```

ii) deflation step

```
simple_deflat:=proc(A,u)
   local p,ap;
   # compute p value
   p:=pindex(u);
   # pth row of A
   ap:=A[p];
   # output
   return A-(u.ap)/u[p];
end proc:
```

Remark: The procedure `pindex` is ill-conditioned (see §2). Thus, it has to be replaced